

# GLOBAL WELL-POSEDNESS AND TWIST-WAVE SOLUTIONS FOR THE INERTIAL QIAN-SHENG MODEL OF LIQUID CRYSTALS

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## Abstract

We consider the inertial Qian-Sheng model of liquid crystals which couples a hyperbolic-type equation involving a second-order material derivative with a forced incompressible Navier-Stokes system. We study the energy law and prove a global well-posedness result. We further provide an example of twist-wave solutions, that is solutions of the coupled system for which the flow vanishes for all times.

## 1. Introduction

The main aim of this article is to study a system describing the hydrodynamics of nematic liquid crystals in the  $Q$ -tensor framework (see for an introduction to the  $Q$ -tensor framework [8],[16]) There exists several such models and we will consider the one proposed by T. Qian and P. Sheng in [12]. As most tensorial models, this one provides an extension of the classical Ericksen-Leslie model [6], in particular capturing the biaxial alignment of the molecules, a feature not available in the classical Ericksen-Leslie model.

Our main interest in this model is due to the fact that it incorporates systematically a certain term that models inertial effects. Details about the physical relevance of this will be provided in the Subsection 1.1, below.

The inertial term is usually neglected on physical grounds, a fact that is also convenient mathematically since keeping it generates considerable analytical and numerical challenges. From a mathematical point of view the system couples a forced incompressible Navier-Stokes system, modelling the flow, with a hyperbolic convection-diffusion system for matrix-valued functions that model the evolution of the orientations of the nematic molecules. The inertial term is responsible for the hyperbolic character of the equation describing the orientation

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of the molecules. This feature is also present in the Ericksen-Leslie model but it is usually neglected in the mathematical studies due to the formidable difficulties in treating it in the presence of the specific unit-length constraint. One can regard our study as a step towards the analytical understanding of the inertial Ericksen-Leslie model where one discards the unit-length constraint.

In order to clearly describe the system it is convenient to introduce some terminology. The local orientation of the molecules is described through a function  $Q$  taking values from  $\Omega \subset \mathbb{R}^d$ , into the set of the so-called  $d$ -dimensional  $Q$ -tensors, that is symmetric and traceless  $d \times d$  matrices:

$$S_0^{(d)} := \{Q \in \mathbb{R}^{d \times d}; Q_{ij} = Q_{ji}, \text{tr}(Q) = 0, i, j = 1, \dots, d\}$$

The evolution of the  $Q$ 's is driven by the free energy of the molecules, as well as the transport, distortion and alignment effects caused by the flow.

The velocity of the centres of masses of molecules obeys a forced incompressible Navier-Stokes system, with an additional stress tensor, a forcing term modelling the effect that the interaction of the molecules has on the dynamics of their centres of masses. Explicitly the equations, in non-dimensional form, are:

$$\begin{aligned} \dot{v} + \nabla p - \frac{\beta_4}{2} \Delta v = & \nabla \cdot \left( -L \nabla Q \odot \nabla Q + \beta_1 Q \text{tr}\{QA\} + \beta_5 A Q + \beta_6 Q A \right) \\ & + \nabla \cdot \left( \frac{\mu_2}{2} (\dot{Q} - [\Omega, Q]) + \mu_1 [Q, (\dot{Q} - [\Omega, Q])] \right) \end{aligned} \quad (1.1)$$

$$\nabla \cdot v = 0 \quad (1.2)$$

$$J \ddot{Q} + \mu_1 \dot{Q} = L \Delta Q - a Q + b(Q^2 - \frac{1}{d} |Q|^2 I_d) - c Q |Q|^2 + \frac{\tilde{\mu}_2}{2} A + \mu_1 [\Omega, Q] \quad (1.3)$$

where  $\dot{f} = (\partial_t + v \cdot \nabla)f$  denotes a *material derivative* and for any two  $d \times d$  matrices  $M, N$ , we denote their commutator as  $[M, N] := MN - NM$ . Furthermore, we denote  $A_{ij} := \frac{1}{2}(v_{i,j} + v_{j,i})$ ,  $\Omega_{ij} := \frac{1}{2}(v_{i,j} - v_{j,i})$ , for  $i, j = 1, \dots, d$ ,  $(\nabla Q \odot \nabla Q)_{ij} := \sum_{k,l=1}^d Q_{kl,i} Q_{kl,j}$  (where for a scalar function  $f$ , we write  $f_{,j}$  for  $\frac{\partial f}{\partial x_j}$ ) and  $|Q| = \sqrt{\text{tr}(Q^2)}$ . The  $I_d$  denotes the  $d \times d$  identity matrix.

The physical relevance of the equations and their meaning is provided in the next subsection, which can be skipped without impeding on the understanding of the remaining mathematical aspects of the paper.

**1.1. Physical aspects.** In the following we consider just the  $d = 3$  case ( out of which one can reduce everything in a standard manner to the  $d = 2$  case) and take the domain  $\Omega$  to be  $\mathbb{R}^3$ . The velocity  $v$  of the centres of masses the molecules satisfies a convection-diffusion fluid-type equation, with forcing provided by the pressure  $p$ , the distortion stress  $\sigma$  and the

viscous stress  $\sigma'$  (here and in the following we use the Einstein summation convention, of summation over repeated indices):

$$\dot{v}_i = (-p\delta_{ij} + \sigma_{ij} + \sigma'_{ij}),_j, \quad (1.4)$$

where  $p$  is the pressure.

The fluid is taken to be incompressible so we have the divergence-free constraint:

$$v_{k,k} = 0. \quad (1.5)$$

The distortion stress  $\sigma$  is given by

$$\sigma_{ij} := -\frac{\partial \mathcal{F}}{\partial (Q_{\alpha\beta,i})} Q_{\alpha\beta,j}$$

where we use the simplest form of the Landau-de Gennes free energy density

$$\mathcal{F}[Q] := \frac{L}{2} |\nabla Q|^2 + \psi_B(Q)$$

modelling the spatial variations through the  $\frac{L}{2} |\nabla Q|^2$  term with positive diffusion coefficient  $L > 0$ , and the nematic ordering enforced through the “bulk term” taken to be of the standard form [8]

$$\psi_B(Q) = \frac{a}{2} \text{tr}(Q^2) - \frac{b}{3} \text{tr}(Q^3) + \frac{c}{4} (\text{tr}(Q^2))^2. \quad (1.6)$$

The viscous stress  $\sigma'$  is given by<sup>1</sup>:

$$\begin{aligned} \sigma'_{ij} := & \beta_1 Q_{ij} Q_{lk} A_{lk} + \beta_4 A_{ij} + \beta_5 Q_{jl} A_{li} + \beta_6 Q_{il} A_{lj} \\ & + \frac{1}{2} \mu_2 \mathcal{N}_{ij} + \mu_1 Q_{il} \mathcal{N}_{lj} - \mu_1 \mathcal{N}_{il} Q_{lj}, \end{aligned} \quad (1.7)$$

where  $\beta_1, \beta_4, \beta_5, \beta_6, \mu_1$  and  $\mu_2$  are viscosity coefficients satisfying the Parodi-type relation

$$\beta_6 - \beta_5 = \mu_2 \quad (1.8)$$

where  $A$  stands for the rate-of-strain tensor

$$A_{ij} = \frac{v_{i,j} + v_{j,i}}{2},$$

i.e. the symmetric part of the velocity gradient, while  $\mathcal{N}$  is the co-rotational time flux of  $Q$ , whose  $(i, j)$ -th component is defined as follows

$$\mathcal{N}_{ij} := (\dot{Q} - \omega \wedge Q + Q \wedge \omega)_{ij} = \partial_t Q_{ij} + v_k Q_{ij,k} - \varepsilon_{ikl} \omega_k Q_{lj} - \varepsilon_{jkl} \omega_k Q_{il}.$$

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<sup>1</sup> Note that in [12] the divergence of a matrix is taken along columns rather than rows as we do in here. However we changed everything consistently to fit our definition of matrix divergence

$\mathcal{N}$  represents the time rate of change of  $Q_{ij}$  with respect to the background fluid angular velocity  $\omega = \frac{1}{2}\nabla \times v$ . Moreover, one can reformulate  $\mathcal{N}$  making use of the vorticity tensor  $\Omega$

$$\Omega_{ij} := \frac{v_{i,j} - v_{j,i}}{2}.$$

Indeed, one can check that

$$\mathcal{N}_{ij} = (\dot{Q} - [\Omega, Q])_{ij} = \dot{Q}_{ij} - \Omega_{il}Q_{lj} + Q_{il}\Omega_{lj},$$

since we have  $\omega \times u = \Omega u$ , for any  $d$ -dimensional vector  $u$ .

The evolution of the order tensor  $Q$  is driven by

$$J\ddot{Q}_{ij} = h_{ij} + h'_{ij} - \lambda\delta_{ij} - \varepsilon_{ijk}\lambda_k. \quad (1.9)$$

where  $\varepsilon_{ijk}$ , the Levi-Civita symbol. The  $\lambda, \lambda_k$  are Lagrange multiplier enforcing the tracelessness and symmetry of the tensor and in our case they can be easily determined as  $\lambda_k = 0$  and  $\lambda = -b\frac{|Q|^2}{3}I_3$  (with  $I_3$  the  $3 \times 3$  identity matrix).

The elastic molecular field  $h$  is

$$h_{ij} := -\frac{\partial \mathcal{F}}{\partial Q_{ij}} + \left( \frac{\partial \mathcal{F}}{\partial (\partial_k Q_{ij,k})} \right)_{,k}$$

and the viscous molecular field  $h'$  is given by:

$$h'_{ij} := \frac{1}{2}\tilde{\mu}_2 A_{ij} - \mu_1 \mathcal{N}_{ij}, \quad (1.10)$$

Let us note that in the paper [12] of Qian and Sheng, the coefficient  $\tilde{\mu}_2 = \mu_2$  but in other papers, such as [10, 11] the coefficient is taken to be  $\tilde{\mu}_2 = -\mu_2$ . The two different choices provide different energy laws as it will be seen in Section 2

We assume all the coefficients to be non-dimensional. For a common physical example, the MBBA material, we have the following relations between the coefficients [13]:

$$\frac{\mu_2}{\mu_1} \sim -1.92, \frac{\beta_1}{\mu_1} \sim 0.17, \frac{\beta_4}{\mu_1} \sim 0.7, \frac{\beta_5}{\mu_1} \sim 0.7, \frac{\beta_6}{\mu_1} \sim -0.79 \quad (1.11)$$

Furthermore, because the coefficient  $\beta_4$  corresponds to the standard Newtonian stress tensor we can assume

$$\beta_4 > 0 \quad (1.12)$$

which fixes the signs for all the viscosities.

The  $J$  in (1.9) stands for the inertial density and it is taken to be greater than 0. This is consistent with the fact that  $J$  has the same sign as the inertia in the Leslie-Ericksen type of model (see Appendix B in [12]) where it is assumed to be positive (see for instance the assumption that J.L. Ericksen makes in [1]).

The inertial term could conceivably play a role when the anisotropic axis is subjected to large accelerations, as motivated by F. Leslie (in the context of the director model) in [7].

Another interesting feature of the inertia is that it captures the wave-like phenomena, and one of the most mysterious and yet simple manifestation of these is related to the so-called twist-waves, introduced by J.L. Ericksen in [1]. These are very special solutions of the coupled system, for which the flow vanishes for all time. The effect of the flow still remains on the  $Q$ -tensor part, by imposing an additional constraint, so these are very special solutions.

**1.2. Main results.** We note first that the system admits a Lyapunov-type functional, up to some relations on the viscosity coefficients. This functional includes the free energy due to the director field, the kinetic energy of the fluid and most importantly the rotational kinetic energy of the director field.

**Theorem 1.1. [Energy law and apriori control of low-regularity norms]**

*We consider the system (1.1),(1.2),(1.3) in  $\mathbb{R}^d$ ,  $d = 2$  or  $d = 3$ . Let us assume that the viscosity coefficients fulfill*

$$\beta_1, \beta_4, \mu_1 \geq 0, \quad (1.13)$$

*and the inertia coefficient  $J$  as well as the diffusion coefficient  $L$  are positive.*

*Moreover, in order to have the free energy of the molecules well defined we assume that the material coefficient  $c$  satisfies*

$$c > 0. \quad (1.14)$$

*Then there exists a constant  $C_d$  depending on  $\tilde{\mu}_2, \beta_5, \beta_6, \mu_2$  such that if the Newtonian viscosity is large enough, i.e.  $\beta_4 > C_d$  then for classical solutions that decay fast enough at infinity<sup>2</sup> the total energy decays, i.e.*

$$\frac{d}{dt} \int_{\mathbb{R}^d} \frac{1}{2} (|v|^2 + J|\dot{Q}|^2 + \frac{L}{2} |\nabla Q|^2) + \psi_B(Q) \, dx \leq 0 \quad (1.15)$$

*Furthermore:*

- *If  $d = 2$  and  $a \geq 0$  then  $\psi_B(Q) \geq 0$  and for any  $T > 0$  we have the apriori bounds:*

$$\begin{aligned} v &\in L^\infty(0, T; L^2(\mathbb{R}^d)) \cap L^2(0, T; H^1(\mathbb{R}^d)), \\ Q &\in L^\infty(0, T; H^1(\mathbb{R}^d)) \quad \text{with} \quad \dot{Q} \in L^\infty(0, T; L^2(\mathbb{R}^d)), \end{aligned} \quad (1.16)$$

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<sup>2</sup>sufficiently fast to be able to integrate by parts in the proof of the theorem, which happens for instance if they are in the function spaces (1.16)

- If  $d = 2$  and  $a < 0$ , or  $d = 3$  (and  $a$  arbitrary) then  $\psi_B(Q)$  can be negative<sup>3</sup>. Then there exists  $\bar{\mu}_1 = \bar{\mu}_1(a, b, c)$ ,  $J_0 := J_0(\bar{\mu}_1, a, b, c)$ , and  $\tilde{C}_d = \tilde{C}_d(\tilde{\mu}_2, \beta_5, \beta_6, \mu_2) > 0$  such that if  $J < J_0$ ,  $\mu_1 > \bar{\mu}_1$ ,  $\beta_4 > \tilde{C}_d$  then the apriori bounds (1.16) hold.

The proof of Theorem 1.1 exhibits the main characteristic feature of the system, that is the mixing of terms that provide the most suitable cancellation of “extraneous” maximal derivatives, i.e. the highest derivatives in  $v$  that appear in the  $Q$  equation and the highest derivatives in  $Q$  that appear in the  $v$  equation.

It is important to observe that despite these apriori estimates, one cannot expect to construct weak solutions just by making use of this energy relation. Indeed, the most common approach in order to construct weak solutions is the compactness method, i.e. construct approximate solutions (satisfying similar apriori bounds) and pass to the limit. In the classical Navier-Stokes equation one needs to take care in dealing with the nonlinear terms, but the apriori estimates available do provide enough control. However, things are much worse in our system (1.1)-(1.3). The main difficulty is inside the stress tensor  $\sigma_{ij}$ , more precisely in the nonlinear term

$$(\nabla Q \odot \nabla Q)_{ij} := \sum_{\alpha, \beta=1}^d Q_{\alpha\beta,i} Q_{\alpha\beta,j} \quad (1.17)$$

Let us note that the estimates provided by the apriori bounds (1.16) do not suffice for passing to the limit in the divergence of (1.17). This is to be contrasted with the case  $J = 0$ . One should keep in mind that a positive inertial density  $J$  leads the order tensor equation to be hyperbolic-like, in contrast to the parabolic structure that occurs when  $J$  is neglected. In the parabolic setting one can make use of regularizing effects, achieving a control on two spatial derivatives of  $Q$  (i.e.  $\Delta Q$ ), which certainly allows to control the limit of a product as in the divergence of (1.17). This feature is lost when  $J$  is positive, so that constructing weak solutions would require a different approach than a rather common compactness one based on estimates (1.16).

Thus one can attempt to construct strong solutions and it will turn out that this can be done. The most interesting aim is then to construct *global in time* solutions. We have been able to obtain them, using the one of main features of the system namely the damping provided by the  $\mu_1$  term. Indeed, if one formally takes the flow  $v$  to be zero in (1.3) then the material derivatives in (1.3) reduce to just time derivatives and the equation becomes a nonlinear damped wave equation. Morally speaking it will be this damping that is responsible for the global existence even in the case when the flow is present. Thus we have:

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<sup>3</sup> and thus the energy decay (1.15) does not suffice for providing the apriori bounds (1.16)

**Theorem 1.2. [Global existence and uniqueness for small initial data]**

Consider the system (1.1),(1.2),(1.3). Let  $\bar{\mu}_1 = \bar{\mu}_1(a, b, c)$ ,  $J_0 := J_0(\bar{\mu}_1, a, b, c)$ , and  $\tilde{C}_d = \tilde{C}_d(\tilde{\mu}_2, \beta_5, \beta_6, \mu_2) > 0$  be as in the last part of Theorem 1.1. We assume that  $J < J_0$ ,  $\mu_1 > \bar{\mu}_1$ ,  $\beta_4 > \tilde{C}_d$  and  $a > 0$ . Furthermore assume  $\beta_1, \mu_1 \geq 0$  and  $J, L > 0$ .

Let  $(v_0, Q_0) : \mathbb{R}^d \rightarrow \mathbb{R}^d \times S_0^{(d)}$  be in  $H^s(\mathbb{R}^d) \times H^{s+1}(\mathbb{R}^d)$  with  $s > \frac{d}{2}$  and  $d = 2$  or  $d = 3$ . Then there exists  $\varepsilon_0 > 0$ , depending on  $s$  and  $d$  such that if

$$\eta_0 := \|v_0\|_{H^s} + \|Q_0\|_{H^{s+1}} + \|\dot{Q}_0\|_{H^s} < \varepsilon_0$$

then there exists a unique strong solution  $(v, Q)$  of (1.4)-(1.9), which is global in time and fulfils

$$\begin{aligned} v &\in L^\infty(\mathbb{R}_+; H^s(\mathbb{R}^2)) \cap L^2(\mathbb{R}_+; H^{s+1}(\mathbb{R}^2)), \\ Q &\in L^\infty(\mathbb{R}_+; H^{s+1}(\mathbb{R}^2)) \quad \text{with} \quad \dot{Q} \in L^\infty(\mathbb{R}_+; H^s(\mathbb{R}^2)). \end{aligned}$$

Moreover, there exists a positive constant  $C$  (independent of the solution) such that

$$\begin{aligned} \|v\|_{L^\infty(\mathbb{R}_+; H^s(\mathbb{R}^2))} + \|\nabla v\|_{L^2(\mathbb{R}_+; H^s(\mathbb{R}^2))} + \|Q\|_{L^\infty(\mathbb{R}_+; H^{s+1}(\mathbb{R}^2))} + \|Q\|_{L^2(\mathbb{R}_+; H^{s+1}(\mathbb{R}^2))} \\ + \|\dot{Q}\|_{L^\infty(\mathbb{R}_+; H^s(\mathbb{R}^2))} + \|\dot{Q}\|_{L^2(\mathbb{R}_+; H^s(\mathbb{R}^2))} \leq C\eta_0. \end{aligned}$$

One important assumption in the above theorem is that the coefficient  $a > 0$ . This captures a regime of physical interest but unfortunately not the most interesting physical regime (which would be for  $a \leq 0$  the “deep nematic” regime, see [8]). Technically this assumption that  $a > 0$  provides a sort of additional “damping in time” that was used previously in related settings in [15], respectively [5].

The difficulties associated with treating the system (1.1),(1.2),(1.3) are generated, as usually (in this type of non-Newtonian fluid) by the forcing term in the Navier-Stokes part. One can essentially think of the system as a highly non-trivial perturbation of a Navier-Stokes system.

However the specific difficulty in our system is that the forcing term involves  $Q$  whose equation is now of hyperbolic nature. This is due to the inertial term that contains a *second order material derivative*. We are not aware of any systematic treatment of such a term in other contexts, but it turns out that the most delicate part of the whole proof is related to its treatment. Indeed, one should start by noticing that this is very far from the case when  $v = 0$  (when it is just  $\partial_t^2 Q$ ) because it is a highly nonlinear operator, for instance for  $v$  and  $Q$  smooth we explicitly have:

$$\ddot{Q} = \partial_t^2 Q + 2(v \cdot \nabla) \partial_t Q + (v_t \cdot \nabla) Q + ((v \cdot \nabla) v \cdot \nabla) Q + v \nabla^2 Q v \quad (1.18)$$

involving an “expensive derivative of  $v$ ” i.e.  $\partial_t v$  and the term  $v \nabla^2 Q v$  that competes in a sense with the regularizing Laplacian  $L \Delta Q$  on the right hand side of the  $Q$ -equation.

Our main trick in dealing with the double material derivative has been to stay as close as possible to the standard cancellation appearing in the context of convective derivatives, which can be formally written as  $\int_{\mathbb{R}^d} (f \cdot \nabla) v f \, dx = 0$  for  $f$  decaying sufficiently fast at infinity.

However, in order to implement this we have to use a higher-order commutator estimate that appears in [2], an estimate which is at the level of homogeneous Sobolev spaces,  $\dot{H}^s$ . This is very convenient for our purposes because the  $H^s(\mathbb{R}^d)$  norm does not allow the cancellation of the worst terms, as in obtaining (1.15) in the  $L^2(\mathbb{R}^d)$ -setting. This difficulty is partially dealt with by reformulating the inner product of  $H^s(\mathbb{R}^d)$  into

$$\langle \omega_1, \omega_2 \rangle_{L^2(\mathbb{R}^2)} + \langle \omega_1, \omega_2 \rangle_{\dot{H}^s(\mathbb{R}^2)} = \int_{\mathbb{R}_\xi^d} (1 + |\xi|^{2s}) \hat{\omega}_1(\xi) \hat{\omega}_2(\xi) d\xi,$$

where  $\dot{H}^s(\mathbb{R}^d)$  stands for the homogeneous Sobolev space with index  $s$ . It is straightforward that this inner product generates the same topology in  $H^s(\mathbb{R}^d)$  with respect to the common one.

Our main work on proving the existence of classical solutions is to obtain an uniform estimate for our approximate solutions, that is to close an estimate of the type:

$$\Phi'(t) + \Psi(t) \leq C\Phi(t)\Psi(t), \quad (1.19)$$

where  $C$  is a suitable positive constant,  $\Phi$  is controlling the  $H^s$ -norms in space for our solution and  $\Psi$  is an integrable-in-time quantity involving  $H^s$ -norms. Then, a rather standard argument (see for instance in the Appendix, Lemma 5.1) allows to propagate the smallness condition on the initial data (i.e. on  $\Phi(0)$ ). This leads the right-hand side of the above equation to be absorbed by the left-hand side, achieving the cited uniform estimates. Finally we construct our classical solution, through a compactness method.

The uniqueness of our solutions is proven by evaluating the difference between two solutions at a regularity level  $s = 0$ , i.e. in  $L^2(\mathbb{R}^2)$ . Our work is mainly to obtain an estimate that leads to the Gronwall lemma. Here the main difficulties are handled taking into account a specific feature of the coupling system related to the difference of the two solutions. This feature allows the cancellation of the worst term when considering certain physically meaningful combination of terms.

It is perhaps interesting to remark that in Theorem 1.2 we do not need consider a positive constant  $c$  in the bulk free energy density  $\psi_B(Q)$ . Usually, this is a necessary condition in order to have  $\psi_B(Q)$  bounded from below the space of  $Q$ -tensors<sup>4</sup>. However we do not need this restriction on  $c$  mainly because we are assuming a smallness condition on the initial data, smallness which will be propagated by the equation.

Finally, one last issue of interest to us are the so-called “*twist waves*”. These are solutions of the coupled system, for which the flow  $v$  is identically zero. The existence of such solutions for the Ericksen-Leslie system was first postulated by J.L. Ericksen in [1], who named them “twist waves” and provided one explicit example.

We note that if  $v = 0$ , the  $Q$ -tensor evolution (1.3) reduces to a nonlinear wave system:

$$JQ_{tt} + \mu_1 Q_t = L\Delta Q - aQ + b(Q^2 - \frac{|Q|^2}{3}I_3) - cQ|Q|^2 \quad (1.20)$$

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<sup>4</sup> also  $c > 0$  is necessary for well-posedness for large data, as can be seen by looking at the  $Q$  equation where we take  $u = 0$  and  $J = 0$



Nevertheless, there is a part of the momentum equation (1.1) that survives as an additional constraint on  $Q$ , namely:

$$\nabla p = \nabla \cdot \left( -\nabla Q \otimes \nabla Q + \frac{\mu_2}{2} Q_t + \mu_1 [Q, Q_t] \right) \quad (1.21)$$

Clearly, because of this additional constraint, only very special types of initial data for (1.20) will generate solutions that respect the constraint (1.21). One example is obtained by taking in  $\mathbb{R}^d$  with  $d = 2$  or  $d = 3$  the ansatz:

$$T(t, x) := f(t, |x|) \bar{H}(x)$$

with  $f : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$  a function to be determined and  $\bar{H}$  the “hedgehog” function (see [3] for details about its physical significance) :

$$\bar{H}_{ij}(x) := \frac{x_i x_j}{|x|^2} - \frac{\delta_{ij}}{d}, i, j = 1, \dots, d$$

Then the (1.20) reduces to:

$$J f_{tt} + \mu_1 f_t = L \left( f_{rr} + f_r \frac{d-1}{r} - \frac{2d}{r^2} f \right) - a f + \frac{b(d-2)}{d} f^2 - \frac{c(d-1)}{d} f^3 \quad (1.22)$$

We can then show:

**Proposition 1.3.** *Let  $f_0 : \mathbb{R} \rightarrow \mathbb{R}$  be a smooth function such that*

$$f(0) = f_r(0) = 0.$$

*Let  $d = 2$  and assume that in (1.22) we have  $J, L, c, \mu_1 > 0$ . Then there exists a function  $f : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ , that is  $C^2$  on  $(0, \infty) \times (0, \infty)$ , and such that both  $f$  and  $f_t$  are smooth functions of  $r$  on  $[0, \infty)$ , solution of the equation (1.22) with  $f(0, r) = f_0(r)$  so that the function  $T(t, x) = f(t, |x|) \bar{H}(x)$  is a smooth twist wave in  $\mathbb{R}^d$ , i.e. a classical solution of (1.20) satisfying the constraints (1.21).*

*Remark 1.4.* A similar statement can be made for  $d = 3$  but for technical reasons we are unable to show the global existence of the twist-wave in this case, but only the local existence in time.

## Organization of the paper

This paper is organised as follows: in the next section we prove Theorem 1.1 concerning the energy law, in Section 3 we present apriori estimates for higher norms and prove the global existence result, namely Theorem 1.2. Finally in Section 4 we construct a specific twist-wave solution, providing the proof of Proposition 1.3 . A number of natural open problems are proposed and discussed in Section 5.

## Notations and conventions

We denote by  $\dot{f}$  the material derivative  $\dot{f} = \partial_t h + v \cdot f$ , where the fluid velocity  $v$  is understood from the context. We also use the Einstein summation convention, that is summation over repeated indices. We denote weighted spaces, by specifying the weighted measure, for instance  $L^2(\mathbb{R}, r^2 dr) = \{f : \mathbb{R} \rightarrow \mathbb{R}, \int_{\mathbb{R}} f^2 r^2 dr < \infty\}$ . We also use the notation  $\|(f, g)\|_X = \|f\|_X + \|g\|_X$  for any elements  $f, g \in X$  with  $X$  a suitable normed space with norm  $\|\cdot\|_X$ . We denote by  $\mathcal{F}$  and  $\mathcal{F}^{-1}$  respectively the Fourier transform and its inverse.

We use  $\varepsilon_{ijk}$ , the Levi-Civita symbol, with indices  $i, j, k$  from 1 to  $d$ , and the comma in the subscript denotes derivative with respect to particular spatial coordinate. If  $M(x)$  is a  $d \times d$ -matrix, then  $\nabla \cdot M$  stands for the vector field  $(M_{ij,j})_{i=1,\dots,d}$ . The  $|M|$  denotes the Frobenius norm of the matrix, i.e.  $|M| = \sqrt{MM^t}$ .

The  $[\cdot, \cdot]$  stands for the usual commutator bracket  $[A, B] := AB - BA$ , for any  $d \times d$ -matrices  $A$  and  $B$ . We denote  $\text{tr}(AB)$  with  $A : B$ . The product  $\nabla A \otimes \nabla B$  is a matrix with  $ij$  component  $(\partial_i A : \partial_j B)$ .

## 2. Energy law and apriori bounds

**Proof of Theorem 1.1:** We multiply the equation (1.1) by  $v_i$  integrate over the space and by parts<sup>5</sup> and use (1.2) to cancel some terms. To the result obtained we add equation (1.3) multiplied by  $\dot{Q}_{ij}$ , integrated over the space and by parts<sup>6</sup> to obtain<sup>7</sup>:

$$\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}^d} \frac{1}{2} (|v|^2 + J|\dot{Q}|^2) dx &= \int_{\mathbb{R}^d} L Q_{kl,j} Q_{kl,i} v_{i,j} + \left( L \Delta Q_{ij} - \mathcal{L} \frac{\partial \psi_B}{\partial Q_{ij}} \right) (\partial_t Q_{ij} + v \cdot \nabla Q_{ij}) dx \\
&\quad - \int_{\mathbb{R}^d} \sigma'_{ij} v_{i,j} dx + \int_{\mathbb{R}^d} h'_{ij} \dot{Q}_{ij} dx \\
&= L \underbrace{\int_{\mathbb{R}^d} Q_{kl,j} Q_{kl,i} v_{i,j} + \Delta Q_{ij} v_k Q_{ij,k} dx}_{:= \mathcal{I}_1} - L \underbrace{\int_{\mathbb{R}^d} \partial_t Q_{ij,k} Q_{ij,k} dx}_{:= \mathcal{I}_2} \\
&\quad - \underbrace{\int_{\mathbb{R}^d} \partial_t Q_{ij} \frac{\partial \psi_B}{\partial Q_{ij}} dx}_{:= \mathcal{I}_3} - \underbrace{\int_{\mathbb{R}^d} v \cdot \nabla Q_{ij} \frac{\partial \psi_B}{\partial Q_{ij}} dx}_{:= \mathcal{I}_4} \\
&\quad - \underbrace{\beta_4 \int_{\mathbb{R}^d} A_{ij} v_{i,j} dx}_{:= \mathcal{I}_5} - \beta_1 \int_{\mathbb{R}^d} Q_{ij} Q_{lk} A_{lk} v_{i,j} dx
\end{aligned} \tag{2.1}$$

<sup>5</sup>without boundary terms, thanks to our assumptions

<sup>6</sup>again without boundary terms thanks to our assumptions

<sup>7</sup> where  $\sigma'$  was defined in (1.7),  $h'$  in (1.10) and  $\psi_B$  in (1.6). The operator  $\mathcal{L}$  denotes the projection onto trace-free matrices.

$$\begin{aligned}
& -\beta_5 \int_{\mathbb{R}^d} A_{il} Q_{lj} v_{i,j} dx - \beta_6 \int_{\mathbb{R}^d} Q_{il} A_{lj} v_{i,j} dx \\
& - \frac{1}{2} \mu_2 \int_{\mathbb{R}^d} \left( \dot{Q}_{ij} - \Omega_{ik} Q_{kj} + Q_{ik} \Omega_{kj} \right) v_{i,j} dx \\
& - \mu_1 \int_{\mathbb{R}^d} \left( Q_{il} \dot{Q}_{lj} - \dot{Q}_{il} Q_{lj} \right) v_{i,j} dx \\
& + \mu_1 \int_{\mathbb{R}^d} \left( Q_{il} [\Omega, Q]_{lj} - [\Omega, Q]_{il} Q_{lj} \right) v_{i,j} dx \\
& + \frac{\tilde{\mu}_2}{2} \int_{\mathbb{R}^d} A_{ij} \dot{Q}_{ij} - \mu_1 \int_{\mathbb{R}^d} \left( \dot{Q}_{ij} - \Omega_{ik} Q_{kj} + Q_{ik} \Omega_{kj} \right) \dot{Q}_{ij} dx
\end{aligned} \tag{2.2}$$

Noting that thanks to (1.2) we have  $\mathcal{J}_1 = \mathcal{J}_4 = 0$  and moving  $\mathcal{J}_2, \mathcal{J}_3, \mathcal{J}_5$  on the left hand side, the last relation becomes

$$\begin{aligned}
& \frac{d}{dt} \int_{\mathbb{R}^d} \frac{1}{2} (|v|^2 + J|\dot{Q}|^2 + L|\nabla Q|^2) + \psi_B(Q) dx + \frac{\beta_4}{2} \int_{\mathbb{R}^d} |\nabla v|^2 dx + \mu_1 \int_{\mathbb{R}^d} |\dot{Q}|^2 dx \\
& = -\beta_1 \int_{\mathbb{R}^d} Q_{ij} Q_{lk} A_{lk} v_{i,j} dx - \beta_5 \int_{\mathbb{R}^d} A_{il} Q_{lj} v_{i,j} dx - \beta_6 \int_{\mathbb{R}^d} Q_{il} A_{lj} v_{i,j} dx \\
& - \frac{1}{2} \mu_2 \int_{\mathbb{R}^d} \left( \dot{Q}_{ij} - \Omega_{ik} Q_{kj} + Q_{ik} \Omega_{kj} \right) v_{i,j} + \frac{\tilde{\mu}_2}{2} \int_{\mathbb{R}^d} A_{ij} \dot{Q}_{ij} \\
& + \mu_1 \int_{\mathbb{R}^d} \left( \dot{Q}_{il} Q_{lj} - Q_{il} \dot{Q}_{lj} \right) v_{i,j} dx + \mu_1 \int_{\mathbb{R}^d} (\Omega_{ik} Q_{kj} - Q_{ik} \Omega_{kj}) \dot{Q}_{ij} dx \\
& - \mu_1 \int_{\mathbb{R}^d} ([\Omega, Q]_{il} Q_{lj} - Q_{il} [\Omega, Q]_{lj}) v_{i,j} dx
\end{aligned} \tag{2.3}$$

Now we analyse each term on the right-hand side of the equality, and we will repeatedly use that  $v_{i,j} = A_{ij} + \Omega_{ij}$  and moreover that  $\text{tr}\{BC\} = B : C$  is null for any  $B$  symmetric and  $C$  skew-adjoint. We begin with

$$\begin{aligned}
-\beta_1 \int_{\mathbb{R}^d} Q_{ij} Q_{lk} A_{lk} v_{i,j} &= -\beta_1 \int_{\mathbb{R}^d} Q_{ij} Q_{lk} A_{lk} A_{ij} - \beta_1 \int_{\mathbb{R}^d} Q_{ij} Q_{lk} A_{lk} \Omega_{ij} \\
&= -\beta_1 \int_{\mathbb{R}^d} (Q : A)^2 - \beta_1 \int_{\mathbb{R}^d} (Q : \Omega)(Q : A) \\
&= -\beta_1 \int_{\mathbb{R}^d} (Q : A)^2,
\end{aligned}$$

observing that  $Q : \Omega = 0$ . Furthermore we have:

$$\begin{aligned}
& -\mu_1 \int_{\mathbb{R}^d} (Q_{il} \dot{Q}_{lj} - \dot{Q}_{il} Q_{lj}) v_{i,j} dx + \mu_1 \int_{\mathbb{R}^d} (\Omega_{ik} Q_{kj} - Q_{ik} \Omega_{kj}) \dot{Q}_{ij} dx = \\
& = \underbrace{\mu_1 \int_{\mathbb{R}^d} \text{tr}\{(Q\dot{Q} - \dot{Q}Q)A\}}_{=0} + \mu_1 \int_{\mathbb{R}^d} \text{tr}\{(Q\dot{Q} - \dot{Q}Q)\Omega\} + \mu_1 \int_{\mathbb{R}^d} \text{tr}\{(\Omega Q - Q\Omega)\dot{Q}\} = \\
& = 2\mu_1 \int_{\mathbb{R}^d} \text{tr}\{[\Omega, Q]\dot{Q}\}.
\end{aligned}$$

Finally

$$\begin{aligned}
& -\mu_1 \int_{\mathbb{R}^d} ([\Omega, Q]_{il} Q_{lj} - Q_{il} [\Omega, Q]_{lj}) v_{i,j} = \mu_1 \int_{\mathbb{R}^d} \text{tr}\{([\Omega, Q]Q - Q[\Omega, Q])\Omega\} = \\
& -\mu_1 \int_{\mathbb{R}^d} \text{tr}\{(\Omega Q - Q\Omega)[\Omega, Q]\} = -\mu_1 \int_{\mathbb{R}^d} |[\Omega, Q]|^2.
\end{aligned}$$

Now we deal with

$$\begin{aligned}
& -\beta_5 \int_{\mathbb{R}^d} A_{il} Q_{lj} v_{i,j} dx - \beta_6 \int_{\mathbb{R}^d} Q_{il} A_{lj} v_{i,j} dx + \frac{\mu_2}{2} \int_{\mathbb{R}^d} (\Omega_{ik} Q_{kj} - Q_{ik} \Omega_{kj}) v_{i,j} dx = \\
& = -\beta_5 \int_{\mathbb{R}^d} \text{tr}\{QA\nabla v\} - \beta_6 \int_{\mathbb{R}^d} \text{tr}\{AQ\nabla v\} + \frac{\mu_2}{2} \int_{\mathbb{R}^d} \text{tr}\{(\Omega Q - Q\Omega)\nabla v\} \\
& = -\beta_5 \int_{\mathbb{R}^d} \text{tr}\{(QA + AQ)\nabla v\} - (\beta_6 - \beta_5) \int_{\mathbb{R}^d} \text{tr}\{AQ\nabla v\} + \frac{\mu_2}{2} \int_{\mathbb{R}^d} \text{tr}\{(\Omega Q - Q\Omega)A\} \\
& = -\beta_5 \int_{\mathbb{R}^d} \text{tr}\{(QA + AQ)A\} - (\beta_6 - \beta_5) \int_{\mathbb{R}^d} \text{tr}\{AQ(A + \Omega)\} + \mu_2 \int_{\mathbb{R}^d} \text{tr}\{AQ\Omega\} \\
& = -2\beta_5 \int_{\mathbb{R}^d} \text{tr}\{AQA\} - (\beta_6 - \beta_5) \int_{\mathbb{R}^d} \text{tr}\{AQA\} - (\beta_6 - \beta_5 + \mu_2) \int_{\mathbb{R}^d} \text{tr}\{AQ\Omega\} \\
& = -(\beta_5 + \beta_6) \int_{\mathbb{R}^d} \text{tr}\{AQA\} - (\beta_6 - \beta_5 + \mu_2) \int_{\mathbb{R}^d} \text{tr}\{AQ\Omega\}.
\end{aligned}$$

We arrange the remaining terms related to  $\mu_2, \tilde{\mu}_2$  as

$$\begin{aligned}
& -\frac{1}{2}\mu_2 \int_{\mathbb{R}^d} \dot{Q}_{ij} v_{i,j} + \frac{\tilde{\mu}_2}{2} \int_{\mathbb{R}^d} A_{ij} \dot{Q}_{ij} = -\frac{\mu_2}{2} \int_{\mathbb{R}^d} \dot{Q} : \nabla v + \frac{\tilde{\mu}_2}{2} \int_{\mathbb{R}^d} A : \dot{Q} \\
& = -\frac{\mu_2}{2} \int_{\mathbb{R}^d} \dot{Q} : A - \underbrace{\frac{\mu_2}{2} \int_{\mathbb{R}^d} \dot{Q} : \Omega}_{=0} + \frac{\tilde{\mu}_2}{2} \int_{\mathbb{R}^d} A : \dot{Q} \\
& = \begin{cases} 0 & \text{if } \tilde{\mu}_2 = \mu_2 \\ -\mu_2 \int_{\mathbb{R}^d} \dot{Q} : A & \text{if } \tilde{\mu}_2 = -\mu_2 \end{cases}
\end{aligned}$$

Summarizing all the previous estimates, we get:

$$\begin{aligned}
& \frac{d}{dt} \int_{\mathbb{R}^d} \frac{1}{2} (|v|^2 + J|\dot{Q}|^2 + L|\nabla Q|^2) + \psi_B(Q) dx + \frac{\beta_4}{2} \int_{\mathbb{R}^d} |\nabla v|^2 dx + \\
& + \beta_1 \int_{\mathbb{R}^d} (Q : A)^2 dx + \mu_1 \int_{\mathbb{R}^d} |\dot{Q} - [\Omega, Q]|^2 dx \\
& = \frac{1}{2} \left( \operatorname{sgn}\left(\frac{\tilde{\mu}_2}{\mu_2}\right) - 1 \right) \mu_2 \int_{\mathbb{R}^d} \dot{Q} : A - (\beta_6 - \beta_5 + \mu_2) \int_{\mathbb{R}^d} \operatorname{tr}\{AQ\Omega\} - (\beta_5 + \beta_6) \int_{\mathbb{R}^d} \operatorname{tr}\{AQA\} \\
& \hspace{25em} (2.4)
\end{aligned}$$

We note out of the above that there exists a constant  $C_d$  depending on  $\tilde{\mu}_2, \beta_5, \beta_6, \mu_2$  such that if  $\beta_4 > C_d$  then the total energy decays, i.e.

$$\frac{d}{dt} \int_{\mathbb{R}^d} \frac{1}{2} (|v|^2 + J|\dot{Q}|^2 + L|\nabla Q|^2) + \psi_B(Q) dx \leq 0 \quad (2.5)$$

If  $d = 2$  then the term  $\operatorname{tr}(Q^3)$  vanishes and if furthermore  $a \geq 0$  then  $\psi_B(Q) \geq 0$  and the previous estimate provides the claimed apriori bounds. However, in general the estimate (2.5) does not suffice for obtaining apriori bounds on the norms of the solutions, because  $\psi_B(Q)$  can be negative. In order to deal with this we need to obtain apriori control on suitable  $L^p$  norms of  $Q$ .

We multiply (1.3) by  $Q$ , take the trace and integrate over space, obtaining:

$$\begin{aligned}
& J \int_{\mathbb{R}^d} \ddot{Q}_{\alpha\beta} Q_{\alpha\beta} dx + \mu_1 \int_{\mathbb{R}^d} \dot{Q}_{\alpha\beta} Q_{\alpha\beta} dx - \underbrace{\mu_1 \int_{\mathbb{R}^d} (\Omega_{\alpha\gamma} Q_{\gamma\beta} - Q_{\alpha\gamma} \Omega_{\gamma\beta}) Q_{\alpha\beta} dx}_{=0} - \\
& - \int_{\mathbb{R}^d} L \Delta Q_{\alpha\beta} Q_{\alpha\beta} dx = \underbrace{\int_{\mathbb{R}^d} (-a Q_{\alpha\beta} Q_{\alpha\beta} + b Q_{\alpha\gamma} Q_{\gamma\beta} Q_{\beta\alpha} - c (Q_{\alpha\beta} Q_{\alpha\beta})^2) dx}_{:=P(Q)} + \\
& \hspace{25em} + \frac{\tilde{\mu}_2}{2} \int_{\mathbb{R}^d} A_{\alpha\beta} Q_{\alpha\beta} dx.
\end{aligned}$$

Now, let us remark that

$$\begin{aligned}
J \int_{\mathbb{R}^d} \ddot{Q}_{\alpha\beta} Q_{\alpha\beta} dx &= J \int_{\mathbb{R}^d} \partial_t \dot{Q}_{\alpha\beta} Q_{\alpha\beta} + v_\gamma \dot{Q}_{\alpha\beta, \gamma} Q_{\alpha\beta} dx \\
&= J \int_{\mathbb{R}^d} \partial_t (\dot{Q}_{\alpha\beta} Q_{\alpha\beta}) + v_\gamma (\dot{Q}_{\alpha\beta} Q_{\alpha\beta})_{, \gamma} dx - J \int_{\mathbb{R}^d} \dot{Q}_{\alpha\beta} \dot{Q}_{\alpha\beta} dx \\
&= J \frac{d}{dt} \int_{\mathbb{R}^d} \frac{1}{2} |\dot{Q} + Q|^2 - \frac{1}{2} |\dot{Q}|^2 - \frac{1}{2} |Q|^2 dx - J \int_{\mathbb{R}^d} |\dot{Q}|^2 dx
\end{aligned}$$

and moreover

$$\begin{aligned} \mu_1 \int_{\mathbb{R}^d} \dot{Q}_{\alpha\beta} Q_{\alpha\beta} dx - L \int_{\mathbb{R}^d} \Delta Q_{\alpha\beta} Q_{\alpha\beta} dx &= \frac{\mu_1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} |Q|^2 dx + \\ &\quad + \underbrace{\mu_1 \int_{\mathbb{R}^d} v_\gamma Q_{\alpha\beta} Q_{\alpha\beta,\gamma} dx}_{=0} + L \int_{\mathbb{R}^d} |\nabla Q|^2 dx. \end{aligned}$$

Thus, summarizing, it turns out that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} J |\dot{Q} + Q|^2 - J |\dot{Q}|^2 + (\mu_1 - J) |Q|^2 dx - J \int_{\mathbb{R}^d} |\dot{Q}|^2 dx + L \int_{\mathbb{R}^d} |\nabla Q|^2 dx &= \\ &= P(Q) + \frac{\tilde{\mu}_2}{2} \int_{\mathbb{R}^d} \text{tr}\{AQ\} dx. \end{aligned} \quad (2.6)$$

We will use this estimate together with (2.4) to obtain an estimate of the  $L^2$  norm of  $Q$  which then will allow to obtain out of (2.4) the desired apriori estimates on  $Q, \dot{Q}$  and  $v$ .

We note that if  $Q$  has eigenvalues  $\lambda, \mu, -\lambda - \mu$  (as it is traceless) we have  $|Q|^2 = 2(\lambda^2 + \mu^2 + \lambda\mu)$  and  $\text{tr}(Q^3) = -3\lambda\mu(\lambda + \mu)$  thus  $\text{tr}(Q^3) \leq \frac{9}{16}|Q|^4 + |Q|^2$ . Furthermore  $\text{tr}(Q^3) = 0$  if  $d = 2$  hence  $Q$  is a two-by-two traceless symmetric matrix. Then there exists  $\bar{\mu}_1 > 0$  depending on  $d, a, b$  and  $c > 0$  such that  $\bar{\mu}_1 |Q|^2 + 4\psi_B(Q) > \varepsilon |Q|^2$ . Then, assuming that  $J < \mu_1$  and adding (2.6) to twice times (2.4) we obtain

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^d} |v|^2 + \frac{J}{2} (|\dot{Q} + Q|^2 + |\dot{Q}|^2) + L |\nabla Q|^2 + \frac{1}{2} (\mu_1 - J) |Q|^2 + 2\psi_B(Q) dx &+ \\ + L \int_{\mathbb{R}^d} |\nabla Q|^2 + 2\beta_1 \int_{\mathbb{R}^d} \text{tr}\{QA\}^2 + 2\beta_4 \int_{\mathbb{R}^d} |\nabla v|^2 dx + 2\mu_1 \int_{\mathbb{R}^d} |\dot{Q} - [\Omega, Q]|^2 dx & \\ = P(Q) + J \int_{\mathbb{R}^d} |\dot{Q}|^2 dx + \frac{\tilde{\mu}_2}{2} \int_{\mathbb{R}^d} \text{tr}\{AQ\} dx + \left( \text{sgn}\left(\frac{\tilde{\mu}_2}{\mu_2}\right) - 1 \right) \mu_2 \int_{\mathbb{R}^d} \text{tr}\{\dot{Q}A\} & \\ - 2(\beta_5 + \beta_6) \int_{\mathbb{R}^d} \text{tr}\{AQA\} - 2(\beta_6 - \beta_5 + \mu_2) \int_{\mathbb{R}^d} \text{tr}\{AQ\Omega\} & \end{aligned} \quad (2.7)$$

Thus for  $\mu_1 > \bar{\mu}_1$ ,  $J$  small enough and  $\beta_4$  large enough we have a Gronwall-type inequality for the  $L^2$  norm of  $Q$  which then can be combined with (2.4) to obtain the apriori bounds (1.16).  $\square$

### 3. Global strong solutions

**3.1. A priori high-norm estimates.** In this subsection we provide the apriori estimates that exhibit in a relatively simple setting the higher-order cancellations and estimates that will allow us to prove afterwards the existence of strong solutions through a suitable approximation scheme, in the next subsection.

We consider the inhomogeneous Sobolev space  $H^s$  with  $s > \frac{d}{2}$ , equipped with inner product

$$\langle u, v \rangle_{H^s} = \langle u, v \rangle_{L^2} + \langle u, v \rangle_{\dot{H}^s}.$$

where

$$\langle u, v \rangle_{\dot{H}^s} := \langle (\sqrt{-\Delta})^s u, (\sqrt{-\Delta})^s v \rangle_{L^2}$$

with  $\sqrt{-\Delta}^s u(\xi) := \mathcal{F}^{-1}(|\xi|^s \mathcal{F}u(\xi))$ .

We recall that for  $s > \frac{d}{2}$  we have  $H^s(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$  and, most importantly for our purposes, it is an algebra, with

$$\|uv\|_{H^s} \leq \|u\|_{H^s} \|v\|_{H^s}.$$

We assume that the solutions are suitably smooth and decaying sufficiently fast at infinity to be able to integrate by parts without boundary terms whenever necessary. Taking the  $H^s$  product between (1.1) and  $v$ , we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|v\|_{H^s}^2 + \beta_4 \|\nabla v\|_{H^s}^2 &= -\langle v \cdot \nabla v, v \rangle_{H^s} + L \langle \nabla Q \odot \nabla Q, \nabla v \rangle_{H^s} - \\ &\quad - \beta_1 \langle \text{tr}\{AQ\}Q, \nabla v \rangle_{H^s} - \beta_5 \langle AQ, \nabla v \rangle_{H^s} - \beta_6 \langle QA, \nabla v \rangle_{H^s} + \\ &\quad - \frac{\mu_2}{2} \langle \dot{Q} - [\Omega, Q], \nabla v \rangle_{H^s} - \mu_1 \langle [Q, \dot{Q}], \nabla v \rangle_{H^s} + \mu_1 \langle [Q, [\Omega, Q]], \nabla v \rangle_{H^s} \end{aligned} \quad (3.1)$$

Now, let us observe that

$$\langle v \cdot \nabla v, v \rangle_{H^s} = \underbrace{\langle v \cdot \nabla v, v \rangle_{L_x^2}}_{=0} + \langle v \cdot \nabla v, v \rangle_{\dot{H}^s}.$$

Since  $s > d/2$ , then  $H^m(\mathbb{R}^d)$  is continuously embedded in  $L^\infty(\mathbb{R}^d)$ , for  $m$  a natural number in  $[s, 1+s)$ . Then, by the classical Gagliardo-Nirenberg inequality we have

$$\begin{aligned} \|v\|_{L^\infty(\mathbb{R}^d)} &\lesssim \|v\|_{L^2(\mathbb{R}^d)}^\theta \|v\|_{\dot{H}^m(\mathbb{R}^d)}^{1-\theta} = \|v\|_{L^2(\mathbb{R}^d)}^\theta \|\nabla v\|_{\dot{H}^{m-1}(\mathbb{R}^d)}^{1-\theta} \\ &\lesssim \|v\|_{L^2(\mathbb{R}^d)}^\theta \|\nabla v\|_{H^{m-1}(\mathbb{R}^d)}^{1-\theta} \lesssim \|v\|_{L^2(\mathbb{R}^d)}^\theta \|\nabla v\|_{H^s(\mathbb{R}^d)}^{1-\theta}, \end{aligned}$$

with  $\theta := \frac{2m-d}{2m}$ . Hence the second term on the right-hand side of the above equality can be estimated as follows:

$$\begin{aligned} |\langle v \cdot \nabla v, v \rangle_{\dot{H}^s}| &= |\langle v \otimes v, \nabla v \rangle_{\dot{H}^s}| \\ &\leq \|v\|_{L_x^\infty} \|v\|_{\dot{H}^s} \|\nabla v\|_{\dot{H}^s} \\ &\lesssim \|v\|_{L_x^2}^\theta \|\nabla v\|_{H^s}^{1-\theta} \|v\|_{\dot{H}^s} \|\nabla v\|_{H^s}, \end{aligned}$$

which yields

$$\begin{aligned} |\langle v \cdot \nabla v, v \rangle_{\dot{H}^s}| &\lesssim \|v\|_{L_x^2}^\theta \|v\|_{\dot{H}^s} \|\nabla v\|_{H^s}^{2-\theta} \\ &\lesssim \|v\|_{L_x^2}^\theta \|v\|_{\dot{H}^s}^{1-\theta} \|v\|_{\dot{H}^s}^\theta \|\nabla v\|_{H^s}^{2-\theta} \\ &\lesssim \|v\|_{H^s} \|v\|_{\dot{H}^s}^\theta \|\nabla v\|_{H^s}^{2-\theta}. \end{aligned} \quad (3.2)$$

Since  $s > d/2 \geq 1$  then  $\|v\|_{\dot{H}^s}^\theta = \|\nabla v\|_{\dot{H}^{s-1}}^\theta \leq \|\nabla v\|_{H^s}^\theta$ , thus

$$|\langle v \cdot \nabla v, v \rangle_{\dot{H}^s}| \leq \|v\|_{H^s} \|\nabla v\|_{H^s}^2 \lesssim \|\nabla v\|_{H^s}^2 \|v\|_{H^s}^2 + c_{\beta_4} \|\nabla v\|_{H^s}^2. \quad (3.3)$$

Now, the second term on the right-hand side of (3.1) is

$$\langle \nabla Q \odot \nabla Q, \nabla v \rangle_{H^s} = \langle \nabla Q \odot \nabla Q, \nabla v \rangle_{L_x^2} + \langle \nabla Q \odot \nabla Q, \nabla v \rangle_{\dot{H}^s}.$$

We will see that  $\langle \nabla Q \odot \nabla Q, \nabla v \rangle_{L_x^2}$  is going to be simplified, while

$$\langle \nabla Q \odot \nabla Q, \nabla v \rangle_{\dot{H}^s} \lesssim \|\nabla Q\|_{L_x^\infty} \|\nabla Q\|_{\dot{H}^s} \|\nabla v\|_{\dot{H}^s} \lesssim \|\nabla Q\|_{H^s}^2 \|\nabla Q\|_{H^s}^2 + c_{\beta_4} \|\nabla v\|_{H^s}^2,$$

Finally, the remaining terms on the right-hand side of (3.1) are controlled as follows:

$$\beta_1 \langle \text{tr}\{AQ\}Q, \nabla v \rangle_{H^s} \lesssim \|A\|_{H^s} \|Q\|_{H^s}^2 \|\nabla v\|_{H^s} \lesssim \|\nabla v\|_{H^s}^2 \|Q\|_{H^s}^2,$$

$$\beta_5 \langle AQ, \nabla v \rangle_{H^s} + \beta_6 \langle QA, \nabla v \rangle_{H^s} \lesssim \|A\|_{H^s} \|Q\|_{H^s} \|\nabla v\|_{H^s} \lesssim \|\nabla v\|_{H^s}^2 \|Q\|_{H^s}^2 + c_{\beta_4} \|\nabla v\|_{H^s}^2,$$

$$\frac{\mu_2}{2} \langle [\Omega, Q], \nabla v \rangle_{H^s} \lesssim \|Q\|_{H^s} \|\nabla v\|_{H^s}^2 \lesssim \|\nabla v\|_{H^s}^2 \|Q\|_{H^s}^2 + c_{\beta_4} \|\nabla v\|_{H^s}^2,$$

$$\mu_1 \langle [Q, \dot{Q}], \nabla v \rangle_{H^s} \lesssim \|Q\|_{H^s} \|\dot{Q}\|_{H^s} \|\nabla v\|_{H^s} \lesssim \|\nabla v\|_{H^s}^2 \|Q\|_{H^s}^2 + c_{\mu_1} \|\dot{Q}\|_{H^s}^2,$$

$$\mu_1 \langle [Q, [\Omega, Q]], \nabla v \rangle_{H^s} \lesssim \|Q\|_{H^s}^2 \|\nabla v\|_{H^s}^2.$$

Thus, summarizing the previous estimates we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|v\|_{H^s}^2 + \beta_4 \|\nabla v\|_{H^s}^2 + \frac{\mu_2}{2} \langle \dot{Q}, \nabla v \rangle_{H^s} - L \langle \nabla Q \odot \nabla Q, \nabla v \rangle_{L_x^2} &\lesssim \\ &\lesssim \|\nabla v\|_{H^s}^2 (\|v\|_{H^s}^2 + \|\nabla Q\|_{H^s}^2 + \|Q\|_{H^s}^2) + c_{\mu_1} \|\dot{Q}\|_{H^s}^2 + c_{\beta_4} \|\nabla v\|_{H^s}^2 + \|\nabla Q\|_{H^s}^2 \|\nabla Q\|_{H^s}^2. \end{aligned} \quad (3.4)$$

Now, let us take the  $H^s$ -inner product between the order equation and  $\dot{Q}$ :

$$\begin{aligned} J \langle \ddot{Q}, \dot{Q} \rangle_{H^s} + \mu_1 \|\dot{Q}\|_{H^s}^2 - \mu_1 \langle [\Omega, Q], \dot{Q} \rangle_{H^s} + \frac{L}{2} \frac{d}{dt} \|\nabla Q\|_{H^s}^2 - L \langle \Delta Q, v \cdot \nabla Q \rangle_{H^s} &= \\ &= -a \langle Q \dot{Q} \rangle_{H^s} + b \langle Q^2, \dot{Q} \rangle_{H^s} - c \langle Q|Q|^2 \rangle_{H^s} + \frac{\tilde{\mu}_2}{2} \langle A, \dot{Q} \rangle_{H^s}. \end{aligned}$$

We begin observing that

$$J \langle \ddot{Q}, \dot{Q} \rangle_{H^s} = \frac{J}{2} \frac{d}{dt} \|\dot{Q}\|_{H^s}^2 + J \langle v \cdot \nabla \dot{Q}, \dot{Q} \rangle_{H^s}$$

and that

$$\begin{aligned} \langle v \cdot \nabla \dot{Q}, \dot{Q} \rangle_{H^s} &= \underbrace{\langle v \cdot \nabla \dot{Q}, \dot{Q} \rangle_{L^2}}_{=0} + \langle v \cdot \nabla \dot{Q}, \dot{Q} \rangle_{\dot{H}^s} \\ &= \langle [(\sqrt{-\Delta})^s, v \cdot \nabla] \dot{Q}, (\sqrt{-\Delta})^s \dot{Q} \rangle_{L^2} + \underbrace{\langle v \cdot \nabla (\sqrt{-\Delta})^s \dot{Q}, (\sqrt{-\Delta})^s \dot{Q} \rangle_{L^2}}_{=0} \\ &\lesssim \|\nabla v\|_{H^s} \|\dot{Q}\|_{H^s}^2 \lesssim \|\nabla v\|_{H^s}^2 \|\dot{Q}\|_{H^s}^2 + c_{\mu_1} \|\dot{Q}\|_{H^s}^2. \end{aligned} \quad (3.5)$$

where for the last inequality we used the commutator estimate from [2]:



$$\|[(\sqrt{-\Delta})^s, v \cdot \nabla]B\|_{L^2} = \|(\sqrt{-\Delta})^s[(v \cdot \nabla)B] - (v \cdot \nabla)(\sqrt{-\Delta})^s B\|_{L_x^2} \leq c\|\nabla v\|_{H^s}\|B\|_{H^s}.$$

Moreover

$$\mu_1 \langle [\Omega, Q], \dot{Q} \rangle_{H^s} \lesssim \|\nabla v\|_{H^s} \|Q\|_{H^s} \|\dot{Q}\|_{H^s} \lesssim \|\nabla v\|_{H^s}^2 \|Q\|_{H^s}^2 + c_{\mu_1} \|\dot{Q}\|_{H^s}^2. \quad (3.6)$$

Now, we have

$$\begin{aligned} \langle \Delta Q, v \cdot \nabla Q \rangle_{\dot{H}^s} &= \langle (\sqrt{-\Delta})^s Q_{\alpha\beta,ii}, (\sqrt{-\Delta})^s (v_j Q_{\alpha\beta,j}) \rangle_{L_x^2} = \\ &= - \langle (\sqrt{-\Delta})^s Q_{\alpha\beta,i}, (\sqrt{-\Delta})^s (v_{j,i} Q_{\alpha\beta,j}) \rangle_{L_x^2} - \langle (\sqrt{-\Delta})^s Q_{\alpha\beta,i}, (\sqrt{-\Delta})^s (v_j Q_{\alpha\beta,ij}) \rangle_{L_x^2} \end{aligned}$$

and

$$\begin{aligned} |\langle (\sqrt{-\Delta})^s Q_{\alpha\beta,i}, (\sqrt{-\Delta})^s (v_{j,i} Q_{\alpha\beta,j}) \rangle_{L_x^2}| &\leq \|\nabla Q\|_{H^s}^2 \|\nabla v\|_{H^s} \lesssim \|\nabla Q\|_{H^s}^2 \|\nabla Q\|_{H^s}^2 + c_{\beta_4} \|\nabla v\|_{H^s}^2, \\ - \langle (\sqrt{-\Delta})^s Q_{\alpha\beta,i}, (\sqrt{-\Delta})^s (v_j Q_{\alpha\beta,ij}) \rangle_{L_x^2} &= - \langle (\sqrt{-\Delta})^s Q_{\alpha\beta,i}, [(\sqrt{-\Delta})^s, v \cdot \nabla] Q_{\alpha\beta,i} \rangle_{L_x^2} - \\ &= - \underbrace{\langle (\sqrt{-\Delta})^s Q_{\alpha\beta,i}, v \cdot \nabla (\sqrt{-\Delta})^s Q_{\alpha\beta,i} \rangle_{L_x^2}}_{=0} \lesssim \|\nabla Q\|_{\dot{H}^s} \|[(\sqrt{-\Delta})^s, v \cdot \nabla] Q_{\alpha\beta,i}\|_{L_x^2} \lesssim \\ &\lesssim \|\nabla Q\|_{H^s}^2 \|\nabla v\|_{H^s} \lesssim \|\nabla Q\|_{H^s}^2 \|\nabla Q\|_{H^s}^2 + c_{\beta_4} \|\nabla v\|_{H^s}^2, \end{aligned}$$

which yields

$$\langle \Delta Q, v \cdot \nabla Q \rangle_{\dot{H}^s} \lesssim \|\nabla Q\|_{H^s}^2 \|\nabla Q\|_{H^s}^2 + c_{\beta_4} \|\nabla v\|_{H^s}^2. \quad (3.7)$$

Finally

$$-a \langle \dot{Q}, Q \rangle_{H^s} = -\frac{a}{2} \frac{d}{dt} \|Q\|_{H^s}^2 - a \langle v \cdot \nabla Q, Q \rangle_{H^s},$$

with

$$|a \langle v \cdot \nabla Q, Q \rangle_{H^s}| \lesssim \|v\|_{H^s} \|\nabla Q\|_{H^s} \|Q\|_{H^s} \lesssim \|Q\|_{H^s}^2 \|v\|_{H^s}^2 + c \|\nabla Q\|_{H^s}^2$$

$$|b \langle \dot{Q}, Q^2 \rangle_{H^s}| \lesssim \|\dot{Q}\|_{H^s} \|Q\|_{H^s}^2 \lesssim \|Q\|_{H^s}^2 \|Q\|_{H^s}^2 + c_{\mu_1} \|\dot{Q}\|_{H^s}^2$$

and

$$|c \langle \dot{Q}, Q|Q|^2 \rangle_{H^s}| \lesssim \|\dot{Q}\|_{H^s} \|Q\|_{H^s}^3 \leq \|Q\|_{H^s}^2 (\|\dot{Q}\|_{H^s}^2 + \|Q\|_{H^s}^2).$$

Thus, summarizing the previous estimates we get

$$\begin{aligned} \frac{d}{dt} \left[ \frac{J}{2} \|\dot{Q}\|_{H^s}^2 + \frac{L}{2} \|\nabla Q\|_{H^s}^2 + \frac{a}{2} \|Q\|_{H^s}^2 \right] &+ \mu_1 \|\dot{Q}\|_{H^s}^2 - L \langle v \cdot \nabla Q, \Delta Q \rangle_{L_x^2} - \frac{\tilde{\mu}_2}{2} \langle A, \dot{Q} \rangle_{H^s} \lesssim \\ &\lesssim (\|Q\|_{H^s}^2 + \|\nabla Q\|_{H^s}^2 + \|\nabla v\|_{H^s}^2 + \|v\|_{H^s}^2) (\|\dot{Q}\|_{H^s}^2 + \|Q\|_{H^s}^2 + \|\nabla Q\|_{H^s}^2) + \\ &+ c_{\beta_4} \|\nabla v\|_{H^s}^2 + c_{\mu_1} \|\dot{Q}\|_{H^s}^2 + c \|\nabla Q\|_{H^s}^2. \end{aligned} \quad (3.8)$$

Now, let us consider the  $H^s$ -inner product between the order tensor equation and  $Q/2$ , namely

$$\begin{aligned} \frac{J}{2} \langle \ddot{Q}, Q \rangle_{H^s} + \frac{\mu_1}{2} \langle \dot{Q}, Q \rangle_{H^s} - \frac{\mu_1}{2} \langle [\Omega, Q], Q \rangle_{H^s} + \frac{L}{2} \|\nabla Q\|_{H^s}^2 + \frac{a}{2} \|Q\|_{H^s}^2 = \\ = \frac{b}{2} \langle Q^2, Q \rangle_{H^s} - \frac{c}{2} \langle Q|Q|^2, Q \rangle_{H^s} + \frac{\tilde{\mu}_2}{4} \langle A, Q \rangle_{H^s}. \end{aligned}$$

First, let us observe that

$$\langle \ddot{Q}, Q \rangle_{H^s} = \langle \ddot{Q}, Q \rangle_{L_x^2} + \langle \ddot{Q}, Q \rangle_{\dot{H}^s}.$$

We have

$$\begin{aligned} \langle \ddot{Q}, Q \rangle_{L_x^2} &= \langle \partial_t \dot{Q}, Q \rangle_{L_x^2} + \langle v \cdot \nabla \dot{Q}, Q \rangle_{L_x^2} \\ &= \frac{d}{dt} \langle \dot{Q}, Q \rangle_{L_x^2} - \langle \dot{Q}, \partial_t Q \rangle_{L_x^2} - \langle \dot{Q}, v \cdot \nabla Q \rangle_{L_x^2} = \frac{d}{dt} \langle \dot{Q}, Q \rangle_{L_x^2} - \|\dot{Q}\|_{L_x^2}^2. \end{aligned}$$

Moreover

$$\langle \partial_t \dot{Q}, Q \rangle_{\dot{H}^s} = \frac{d}{dt} \langle \dot{Q}, Q \rangle_{\dot{H}^s} - \langle \dot{Q}, \partial_t Q \rangle_{\dot{H}^s}$$

and

$$\begin{aligned} \langle v \cdot \nabla \dot{Q}, Q \rangle_{\dot{H}^s} &= \langle (\sqrt{-\Delta})^s (v \cdot \nabla \dot{Q}), (\sqrt{-\Delta})^s Q \rangle_{L_x^2} \\ &= \langle [(\sqrt{-\Delta})^s, v \cdot \nabla] \dot{Q}, (\sqrt{-\Delta})^s Q \rangle_{L_x^2} - \langle (\sqrt{-\Delta})^s \dot{Q}, v \cdot \nabla (\sqrt{-\Delta})^s Q \rangle_{L_x^2} \quad (3.9) \\ &= \langle [(\sqrt{-\Delta})^s, v \cdot \nabla] \dot{Q}, (\sqrt{-\Delta})^s Q \rangle_{L_x^2} + \\ &\quad + \langle (\sqrt{-\Delta})^s \dot{Q}, [(\sqrt{-\Delta})^s, v \cdot \nabla] Q \rangle_{L_x^2} - \langle \dot{Q}, v \cdot \nabla Q \rangle_{\dot{H}^s}, \end{aligned}$$

Thus, summarizing, we get

$$\begin{aligned} \frac{J}{2} \langle \ddot{Q}, Q \rangle_{H_x^s} &= \frac{J}{2} \frac{d}{dt} \langle \dot{Q}, Q \rangle_{H^s} - \frac{J}{2} \|\dot{Q}\|_{H^s}^2 + \\ &\quad + \frac{J}{2} \langle [(\sqrt{-\Delta})^s, v \cdot \nabla] \dot{Q}, (\sqrt{-\Delta})^s Q \rangle_{L_x^2} + \frac{J}{2} \langle (\sqrt{-\Delta})^s \dot{Q}, [(\sqrt{-\Delta})^s, v \cdot \nabla] Q \rangle_{L_x^2} \end{aligned} \quad (3.10)$$

with the estimate

$$\begin{aligned} \frac{J}{2} \langle [(\sqrt{-\Delta})^s, v \cdot \nabla] \dot{Q}, (\sqrt{-\Delta})^s Q \rangle_{L_x^2} + \frac{J}{2} \langle (\sqrt{-\Delta})^s \dot{Q}, [(\sqrt{-\Delta})^s, v \cdot \nabla] Q \rangle_{L_x^2} \\ \lesssim \|\nabla v\|_{H^s} \|\dot{Q}\|_{H^s} \|Q\|_{H^s} \lesssim \|\nabla v\|_{H^s}^2 \|Q\|_{H^s}^2 + c_{\mu_1} \|\dot{Q}\|_{H^s}^2. \end{aligned} \quad (3.11)$$

Furthermore

$$\frac{J}{2} \frac{d}{dt} \langle \dot{Q}, Q \rangle_{\dot{H}^s} = \frac{J}{4} \frac{d}{dt} [\|\dot{Q} + Q\|_{\dot{H}^s}^2 - \|\dot{Q}\|_{\dot{H}^s}^2 - \|Q\|_{\dot{H}^s}^2]. \quad (3.12)$$

On the other hand

$$\frac{\mu_1}{2} \langle \dot{Q}, Q \rangle_{H^s} = \frac{\mu_1}{2} \langle \partial_t Q, Q \rangle_{H^s} + \frac{\mu_1}{2} \langle v \cdot \nabla Q, Q \rangle_{H^s} = \frac{\mu_1}{4} \frac{d}{dt} \|Q\|_{H^s}^2 + \frac{\mu_1}{2} \langle v \cdot \nabla Q, Q \rangle_{H^s},$$

with

$$\frac{\mu_1}{2} \langle v \cdot \nabla Q, Q \rangle_{H^s} \lesssim \|v\|_{H^s} \|\nabla Q\|_{H^s} \|Q\|_{H^s} \lesssim \|Q\|_{H^s}^2 \|v\|_{H^s}^2 + c \|\nabla Q\|_{H^s}^2.$$

Then

$$\frac{\mu_1}{2} \langle [\Omega, Q], Q \rangle_{H^s} \lesssim \|\nabla v\|_{H^s} \|Q\|_{H^s}^2 \lesssim \|Q\|_{H^s}^2 \|Q\|_{H^s}^2 + c_{\beta_4} \|\nabla v\|_{H^s}^2,$$

$$\left| \frac{b}{2} \langle Q^2, Q \rangle_{H^s} \right| \lesssim \|Q\|_{H^s}^3 \lesssim \|Q\|_{H^s}^2 \|Q\|_{H^s}^2 + c_a \|Q\|_{H^s}^2,$$

$$\left| \frac{c}{2} \langle Q|Q|^2, Q \rangle_{H^s} \right| \lesssim \|Q\|_{H^s}^2 \|Q\|_{H^s}^2$$

and finally

$$\left| \frac{\tilde{\mu}_2}{4} \langle A, Q \rangle_{H^s} \right| \leq \left| \frac{\tilde{\mu}_2}{4} \right| \|\nabla v\|_{H^s} \|Q\|_{H^s} \leq \frac{|\tilde{\mu}_2|^2}{32a(1-\varepsilon)} \|\nabla v\|_{H^s}^2 + \frac{a}{2} (1-\varepsilon) \|Q\|_{H^s}^2$$

Then, summarizing the previous estimates, we get

$$\begin{aligned} & \frac{d}{dt} \left[ \frac{J}{4} \|\dot{Q} + Q\|_{H^s}^2 - \frac{J}{4} \|\dot{Q}\|_{H^s}^2 + \frac{\mu_1 - J}{4} \|Q\|_{H^s}^2 \right] - \frac{J}{2} \|\dot{Q}\|_{H^s}^2 + \frac{a}{2} \varepsilon \|Q\|_{H^s}^2 - \\ & - \frac{|\tilde{\mu}_2|^2}{32a(1-\varepsilon)} \|\nabla v\|_{H^s}^2 + \frac{L}{2} \|\nabla Q\|_{H^s}^2 \lesssim \left( \|\nabla v\|_{H^s}^2 + \|Q\|_{H^s}^2 \right) \left( \|Q\|_{H^s}^2 + \|v\|_{H^s}^2 \right) + \\ & + c_{\beta_4} \|\nabla v\|_{H^s}^2 + c_{\mu_1} \|\dot{Q}\|_{H^s}^2 + c_a \|Q\|_{H^s}^2 + c \|\nabla Q\|_{H^s}^2. \end{aligned} \quad (3.13)$$

Finally, taking the sum between (3.4), (3.8) and (3.13) and assuming  $c_{\beta_4}$ ,  $c_{\mu_1}$ ,  $c$  and  $c_a$  small enough, we get for a suitable  $\delta > 0$  small enough:

$$\begin{aligned} & \frac{d}{dt} \left[ \frac{1}{2} \|v\|_{H^s}^2 + \frac{J}{4} \|Q + \dot{Q}\|_{H^s}^2 + \frac{J}{4} \|\dot{Q}\|_{H^s}^2 + \left( \frac{a}{2} + \mu_1 - \frac{J}{4} \right) \|Q\|_{H^s}^2 + \frac{L}{2} \|\nabla Q\|_{H^s}^2 \right] + \\ & + \left( \beta_4 - \frac{|\tilde{\mu}_2|^2}{32(1-\varepsilon)a} - \delta \right) \|\nabla v\|_{H^s}^2 + \left( \mu_1 - \frac{J}{2} - \delta \right) \|\dot{Q}\|_{H^s}^2 + \left( \frac{a}{2} \varepsilon - \delta \right) \|Q\|_{H^s}^2 + \left( \frac{L}{2} - \delta \right) \|\nabla Q\|_{H^s}^2 \\ & \lesssim \left( \|\nabla v\|_{H^s}^2 + \|\dot{Q}\|_{H^s}^2 + \|Q\|_{H^s}^2 + \|\nabla Q\|_{H^s}^2 \right) \left( \|v\|_{H^s}^2 + \|Q\|_{H^s}^2 + \|\dot{Q}\|_{H^s}^2 + \|\nabla Q\|_{H^s}^2 \right), \end{aligned}$$

where we have used

$$\langle \nabla Q \odot \nabla Q, \nabla v \rangle_{L_x^2} + \langle v \cdot \nabla Q, \Delta Q \rangle_{L_x^2} = 0.$$

The last estimate allows, under suitable relations on the coefficients, to obtain a differential inequality of the type (1.19) with  $\Phi(t) := \|v\|_{H^s}^2 + \|Q\|_{H^s}^2 + \|\dot{Q}\|_{H^s}^2 + \|\nabla Q\|_{H^s}^2$  and  $\Psi(t) := \|\nabla v\|_{H^s}^2 + \|\dot{Q}\|_{H^s}^2 + \|Q\|_{H^s}^2 + \|\nabla Q\|_{H^s}^2$  which allows to control apriori these norms globally in time, for small data (see also the Lemma 5.1 in the Appendix).

**3.2. Strong solutions. Proof of Theorem 1.2:** We divide the proof into the existence and uniqueness parts. The existence is based on a Friedrichs-type scheme that preserves the structure exhibited in the higher-order energy laws, which allows to construct approximate solutions. The uniqueness is achieved afterwards through to an  $H^s$ -type energy estimate.

**Existence part:** In order to construct global strong solutions, we use the classical Friedrichs scheme and obtain estimates similar to the ones in the previous section. We define the mollifying operator

$$J_n f(\xi) := \mathcal{F}^{-1} \left( 1_{\{2^{-n} \leq |\xi| \leq 2^n\}} \mathcal{F} f \right),$$

The approximate momentum equation then reads as follows:

$$\begin{aligned} J_n \partial_t v^{(n)} + \mathcal{P} J_n (J_n v^{(n)} \cdot \nabla J_n v^{(n)}) + \frac{\beta_4}{2} \Delta J_n v^{(n)} = & -L \nabla \cdot \mathcal{P} \left\{ J_n (\nabla J_n Q^{(n)} \otimes \nabla J_n Q^{(n)}) \right\} + \\ & + \nabla \cdot \mathcal{P} \left\{ \beta_1 J_n (J_n Q^{(n)} \operatorname{tr} \{ (J_n Q^{(n)} J_n A^{(n)}) \}) + \beta_5 J_n (J_n A^{(n)} J_n Q^{(n)}) + \beta_6 J_n (J_n Q^{(n)} J_n A^{(n)}) \right\} \\ & + \nabla \cdot \mathcal{P} \left\{ \frac{\mu_2}{2} (J_n \dot{Q}^{(n)} - J_n [J_n \Omega^{(n)}, J_n Q^{(n)}]) + \mu_1 J_n [J_n Q^{(n)}, (J_n \dot{Q}^{(n)} - [J_n \Omega^{(n)}, J_n Q^{(n)}])] \right\}, \end{aligned} \quad (3.14)$$

where  $\mathcal{P}$  denotes the Leray projector onto divergence-free vector fields, and we denote

$$\Omega_{ij}^{(n)} := \frac{v_{i,j}^{(n)} - v_{j,i}^{(n)}}{2}, \quad A_{ij}^{(n)} := \frac{v_{i,j}^{(n)} + v_{j,i}^{(n)}}{2}, \quad i, j = 1, \dots, d$$

Similarly, the approximate order tensor equation is

$$\begin{aligned} J J_n \ddot{Q}^{(n)} + \mu_1 J_n \dot{Q}^{(n)} = & L \Delta J_n Q^{(n)} - a J_n Q^{(n)} + b J_n (J_n Q^{(n)} J_n Q^{(n)}) \\ & - b \operatorname{tr} \{ J_n (J_n Q^{(n)} J_n Q^{(n)}) \} \frac{\operatorname{Id}}{d} + c J_n \left( J_n Q^{(n)} \operatorname{tr} \{ (J_n Q^{(n)} J_n Q^{(n)}) \} \right) \\ & + \frac{\tilde{\mu}_2}{2} J_n A^{(n)} + \mu_1 J_n [J_n \Omega^{(n)}, J_n Q^{(n)}] \end{aligned}$$

where we have used the abuse of notation

$$\dot{f}^{(n)} := \partial_t f^{(n)} + J_n (J_n v^{(n)} \cdot \nabla J_n f^{(n)}).$$

The system above can be regarded as an ordinary differential equation in  $L^2$  verifying the conditions of the Cauchy-Lipschitz theorem. Thus it admits a unique maximal solution  $(v^{(n)}, Q^{(n)})$  in  $C^1([0, T^n], L^2)$ . As we have  $(\mathcal{P} J_n)^2 = \mathcal{P} J_n$  and  $J_n^2 = J_n$ , the pair  $(J_n v^{(n)}, J_n Q^{(n)})$  is also a solution of the previous system. Hence, by uniqueness we get that  $(J_n v^{(n)}, J_n Q^{(n)}) = (v^{(n)}, Q^{(n)})$ , and moreover the pair  $(v^{(n)}, Q^{(n)})$  belongs to  $C^1([0, T^n], H^\infty)$

and solves the system

$$\left\{ \begin{array}{l} \partial_t v^{(n)} + \mathcal{P} J_n(v^{(n)} \cdot \nabla v^{(n)}) + \frac{\beta_4}{2} \Delta v^{(n)} = -L \nabla \cdot \mathcal{P} \left\{ J_n(\nabla Q^{(n)} \otimes \nabla Q^{(n)}) \right\} + \\ + \nabla \cdot \mathcal{P} \left\{ \beta_1 J_n(Q^{(n)} \text{tr}\{Q^{(n)} A^{(n)}\}) + \beta_5 J_n(A^{(n)} Q^{(n)}) + \beta_6 J_n(Q^{(n)} A^{(n)}) \right\} + \\ + \nabla \cdot \mathcal{P} \left\{ \frac{\mu_2}{2} (\dot{Q}^{(n)} - J_n[\Omega^{(n)}, Q^{(n)}]) + \mu_1 J_n[Q^{(n)}, (\dot{Q}^{(n)} - [\Omega^{(n)}, Q^{(n)}])] \right\} \right\}, \\ J \ddot{Q}^{(n)} + \mu_1 \dot{Q}^{(n)} = L \Delta Q^{(n)} - a Q^{(n)} + b J_n(Q^{(n)} Q^{(n)}) \\ - b \text{tr}\{J_n(Q^{(n)} Q^{(n)})\} \frac{\text{Id}}{d} + c J_n(Q^{(n)} \text{tr}\{(Q^{(n)} Q^{(n)})\}) \\ + \frac{\tilde{\mu}_2}{2} A^{(n)} + \mu_1 J_n[\Omega^{(n)}, Q^{(n)}] \end{array} \right. \quad (3.15)$$

Arguing similarly as in the proof of the apriori estimates and taking advantage of the fact that  $J_n$  is a self-adjoint operator in  $L^2$ , in order to obtain similar cancellations we get (for a  $\delta > 0$  suitably small):

$$\begin{aligned} & \frac{d}{dt} \left[ \frac{1}{2} \|v^{(n)}\|_{H^s}^2 + \frac{J}{4} \|Q^{(n)} + \dot{Q}^{(n)}\|_{H^s}^2 + \frac{J}{4} \|\dot{Q}^{(n)}\|_{H^s}^2 + \left(\frac{a}{2} + \mu_1 - \frac{J}{4}\right) \|Q^{(n)}\|_{H^s}^2 + \frac{L}{2} \|\nabla Q^{(n)}\|_{H^s}^2 \right] \\ & + \left(\beta_4 - \frac{|\tilde{\mu}_2|^2}{32(1-\varepsilon)a} - \delta\right) \|\nabla v^{(n)}\|_{H^s}^2 + \left(\mu_1 - \frac{J}{2} - \delta\right) \|\dot{Q}^{(n)}\|_{H^s}^2 + \left(\frac{a}{2}\varepsilon - \delta\right) \|Q^{(n)}\|_{H^s}^2 + \left(\frac{L}{2} - \delta\right) \|\nabla Q^{(n)}\|_{H^s}^2 \\ & \lesssim \left( \|\nabla v^{(n)}\|_{H^s}^2 + \|\dot{Q}^{(n)}\|_{H^s}^2 + \|Q^{(n)}\|_{H^s}^2 + \|\nabla Q^{(n)}\|_{H^s}^2 \right) \times \\ & \quad \times \left( \|v^{(n)}\|_{H^s}^2 + \|Q^{(n)}\|_{H^s}^2 + \|\dot{Q}^{(n)}\|_{H^s}^2 + \|\nabla Q^{(n)}\|_{H^s}^2 \right), \end{aligned} \quad (3.16)$$

Defining the functions  $x(t)$  and  $y(t)$  by

$$\begin{aligned} x(t) &:= \|\nabla v^{(n)}\|_{H^s}^2 + \|\dot{Q}^{(n)}\|_{H^s}^2 + \|Q^{(n)}\|_{H^s}^2 + \|\nabla Q^{(n)}\|_{H^s}^2, \\ y(t) &:= \|v^{(n)}\|_{H^s}^2 + \|Q^{(n)}\|_{H^s}^2 + \|\dot{Q}^{(n)}\|_{H^s}^2 + \|\nabla Q^{(n)}\|_{H^s}^2, \end{aligned}$$

respectively, and thanks to Lemma 5.1 and inequality (3.16), we get the following bound:

$$\begin{aligned} & \sup_{t \in \mathbb{R}_+} \left\{ \|v^{(n)}(t)\|_{H^s}^2 + \|Q^{(n)}(t)\|_{H^s}^2 + \|\dot{Q}^{(n)}(t)\|_{H^s}^2 + \|\nabla Q^{(n)}(t)\|_{H^s}^2 \right\} + \\ & + \int_{\mathbb{R}_+} \left\{ \|\nabla v^{(n)}(t)\|_{H^s}^2 + \|\dot{Q}^{(n)}(t)\|_{H^s}^2 + \|Q^{(n)}(t)\|_{H^s}^2 + \|\nabla Q^{(n)}(t)\|_{H^s}^2 \right\} dt \\ & \lesssim \|v_0\|_{H^s}^2 + \|Q_0\|_{H^s}^2 + \|\dot{Q}_0\|_{H^s}^2 + \|\nabla Q_0\|_{H^s}^2. \end{aligned}$$

We claim that these uniform estimates allow us to pass to the limit as,  $n$  goes to  $\infty$ . We first observe that we can obtain a uniform bound also for  $\partial_t Q^n$  in  $L_t^\infty H^s$ . Indeed

$$\begin{aligned} \sup_{t \in \mathbb{R}_+} \|\partial_t Q^n\|_{H^s} &= \sup_{t \in \mathbb{R}_+} \|\dot{Q}^n - v^{(n)} \cdot \nabla Q^{(n)}\|_{H^s} \leq \|Q^n\|_{L_t^\infty H^s} + \|v^n\|_{L_t^\infty H^s} \|\nabla Q^n\|_{L_t^\infty H^s} \\ &\lesssim \|v_0\|_{H^s}^2 + \|Q_0\|_{H^s}^2 + \|\dot{Q}_0\|_{H^s}^2 + \|\nabla Q_0\|_{H^s}^2. \end{aligned}$$

Thus, by classical compactness, weak convergence arguments and thanks to the Aubin-Lions lemma, there exists

$$Q \in L_t^\infty H^{s+1} \cap L_t^2 H^{s+1}, \quad v \in L_t^\infty H^s \cap L_t^2 H^{s+1}, \quad \text{and} \quad \omega \in L_t^\infty H^s \cap L_t^2 H^s,$$

such that, up to a subsequence, we have the following convergences

$$\begin{aligned} Q^{(n)} &\rightarrow Q \quad \text{strong in } L_{t,loc}^\infty H_{loc}^{s+1-\mu} \\ \dot{Q}^{(n)} &\rightarrow \omega \quad \text{strong in } L_{t,loc}^\infty H_{loc}^{s-\mu} \\ v^{(n)} &\rightarrow v \quad \text{strong in } L_{t,loc}^\infty H_{loc}^{s-\mu} \\ \nabla v^{(n)} &\rightharpoonup \nabla v \quad \text{weak in } L_t^2 H^s \end{aligned}$$

for any suitably small positive constant  $\mu$ .

Assuming  $s - \mu > d/2$ , we have that  $J_n(v^{(n)} \cdot \nabla Q^{(n)})$  strongly converges to  $v \cdot \nabla Q$  in  $L_{t,loc}^\infty H_{loc}^{s-\mu}$ , as  $n \rightarrow \infty$ , with  $v \cdot \nabla Q \in L_t^\infty H^s$ . Furthermore

$$\partial_t Q = \lim_{n \rightarrow \infty} \partial_t Q^{(n)} = \lim_{n \rightarrow \infty} \left( \dot{Q}^{(n)} - v^{(n)} \cdot \nabla Q^{(n)} \right) = \omega - v \cdot \nabla Q \in L_t^\infty H^s,$$

where the limits are considered in the distributional sense. Then, we deduce  $\partial_t Q \in L_t^\infty H^s$  and  $\omega = \dot{Q} \in L_t^\infty H^s$ . Finally, the order-tensor equation yields

$$\begin{aligned} J \partial_t \dot{Q}^{(n)} &= -J J_n(v^{(n)} \cdot \nabla \dot{Q}^{(n)}) - \mu_1 \dot{Q}^{(n)} + \mu_1 J_n[\Omega^{(n)}, Q^{(n)}] + L \Delta Q^{(n)} + \frac{\tilde{\mu}_2}{2} A^{(n)} - \\ &\quad - a Q^{(n)} + b \left( J_n(Q^{(n)} Q^{(n)}) + \text{tr}\{(Q^{(n)} Q^{(n)})\} \frac{\text{Id}}{d} \right) - c J_n(Q^{(n)} \text{tr}\{(Q^{(n)} Q^{(n)})\}), \end{aligned}$$

hence, observing that

$$\begin{aligned} \|J_n(v^{(n)} \cdot \nabla \dot{Q}^{(n)})\|_{H^{s-1}} &\lesssim \|v^{(n)} \cdot \nabla \dot{Q}^{(n)}\|_{H^{s-1}} \\ &\lesssim \|\nabla \cdot \{v^{(n)} \otimes \dot{Q}^{(n)}\}\|_{H^{s-1}} \\ &\lesssim \|v^{(n)} \otimes \dot{Q}^{(n)}\|_{H^s} \lesssim \|v^{(n)}\|_{H^s} \|\dot{Q}^{(n)}\|_{H^s}, \end{aligned}$$

then  $\partial_t \dot{Q}^{(n)}$  belongs to  $L_{t,loc}^1 H^{s-1}$ , with uniformly in  $n$  bounded seminorms. Thus

$$\partial_t \dot{Q}^{(n)} \rightharpoonup \partial_t \dot{Q} \quad \text{weakly in } L_{t,loc}^1 H^{s-1},$$

up to a subsequence. Moreover, since  $J_n(v^{(n)} \otimes \dot{Q}^{(n)})$  weakly converges to  $v \otimes \dot{Q}$  in  $L^1_{t,loc} H^s$ , then  $J_n(v^{(n)} \cdot \nabla \dot{Q}^{(n)})$  weakly converges to  $v \cdot \nabla \dot{Q}$  in  $L^1_{t,loc} H^s$ . Then, summarizing we deduce that  $\ddot{Q}^{(n)}$  weakly converges to  $\ddot{Q}$  in  $L^1_{t,loc} H^s$ .

These convergences allow us to pass to the limit in the classical solutions of (3.15), deducing that  $(u, Q)$  is classical solution of system (1.1) and (1.3).

**Uniqueness part:** We now prove the uniqueness of the strong solutions previously obtained. Let us consider  $(u_1, Q_1)$  and  $(u_2, Q_2)$  to be strong solutions with same initial data. From here on we will use the following notation:

$$\delta Q := Q_1 - Q_2, \quad \delta \dot{Q} := \dot{Q}_1 - \dot{Q}_2, \quad \delta v := v_1 - v_2, \quad \delta A := A_1 - A_2, \quad \delta \Omega := \Omega_1 - \Omega_2.$$

We begin the proof by considering the difference between the order-parameter equations of the two solutions, namely

$$\begin{aligned} J \left[ (\delta \dot{Q})_t + v_1 \cdot \nabla \delta \dot{Q} + \delta v \cdot \nabla \dot{Q}_2 \right] + \mu_1 \delta \dot{Q} &= L \Delta \delta Q - a \delta Q + b [Q_1 \delta Q + \delta Q Q_2 + \\ &+ \text{tr}\{Q_1 \delta Q + \delta Q Q_2\} \frac{\text{Id}}{d}] - c \delta Q \text{tr}\{Q_1^2\} - c Q_2 \text{tr}\{\delta Q Q_1\} - c Q_2 \text{tr}\{Q_2 \delta Q\} + \\ &\frac{\tilde{\mu}_2}{2} \delta A + \mu_1 [\Omega_1, \delta Q] + \mu_1 [\delta \Omega, Q_2]. \end{aligned}$$

We multiply both the left and right-hand sides by  $\delta \dot{Q}$ , integrating over  $\mathbb{R}^d$ ,

$$\begin{aligned} \frac{d}{dt} \left[ \frac{J}{2} \|\delta \dot{Q}\|_{L_x^2}^2 + \frac{L}{2} \|\nabla \delta Q\|_{L_x^2}^2 + \frac{a}{2} \|\delta Q\|_{L_x^2}^2 \right] + \mu_1 \|\delta \dot{Q}\|_{L_x^2}^2 &= L \langle \Delta \delta Q, v_1 \cdot \nabla \delta Q \rangle_{L_x^2} + \\ &+ L \langle \Delta \delta Q, \delta v \cdot \nabla Q_2 \rangle_{L_x^2} - J \langle v_1 \cdot \nabla \delta \dot{Q}, \delta \dot{Q} \rangle_{L_x^2} - J \langle \delta v \cdot \nabla \dot{Q}_2, \delta \dot{Q} \rangle_{L_x^2} - \\ &- a \langle \delta Q, v_1 \cdot \nabla \delta Q + \delta v \cdot \nabla Q_2 \rangle_{L_x^2} + b \langle Q_1 \delta Q + \delta Q Q_2, \delta \dot{Q} \rangle_{L_x^2} - \\ &- c \langle \delta Q \text{tr}\{Q_1^2\} + Q_2 \text{tr}\{\delta Q Q_2\} + Q_2 \text{tr}\{Q_2 \delta Q\}, \delta \dot{Q} \rangle_{L_x^2} + \\ &+ \frac{\tilde{\mu}_2}{2} \langle \delta A, \delta \dot{Q} \rangle_{L_x^2} + \mu_1 \langle [\Omega_1, \delta Q] + [\delta \Omega, Q_2], \delta \dot{Q} \rangle_{L_x^2}. \end{aligned} \quad (3.17)$$

We now estimate each term on the right-hand side. First we remark that

$$\begin{aligned} \langle \Delta \delta Q, v_1 \cdot \nabla \delta Q \rangle_{L_x^2} &= \langle \delta Q_{\alpha\beta, jj}, (v_1)_i \delta Q_{\alpha\beta, i} \rangle_{L_x^2} \\ &= - \langle \delta Q_{\alpha\beta, j}, (v_1)_{i,j} \delta Q_{\alpha\beta, i} \rangle_{L_x^2} - \underbrace{\langle \delta Q_{\alpha\beta, j}, (v_1)_i \delta Q_{\alpha\beta, ij} \rangle_{L_x^2}}_{=0}, \end{aligned}$$

where for the second equality we have integrated by parts. Then we obtain

$$\langle \Delta \delta Q, v_1 \cdot \nabla \delta Q \rangle_{L_x^2} \lesssim \|\nabla \delta Q\|_{L_x^2} \|\nabla \delta Q\|_{L_x^2} \|\nabla v_1\|_{L_x^\infty} \lesssim \|\nabla v_1\|_{H^s} \|\nabla \delta Q\|_{L_x^2}^2,$$

Similarly, we can proceed integrating by parts also for the second term, namely

$$\begin{aligned} \langle \Delta \delta Q, \delta v \cdot \nabla Q_2 \rangle_{L_x^2} &= \langle \delta Q_{\alpha\beta,jj}, \delta v_i \cdot (Q_2)_{\alpha\beta,i} \rangle_{L_x^2} \\ &= - \underbrace{\langle \delta Q_{\alpha\beta,j}, \delta v_{i,j} \cdot (Q_2)_{\alpha\beta,i} \rangle_{L_x^2}}_{\mathcal{A}} - \underbrace{\langle \delta Q_{\alpha\beta,j}, \delta v_i \cdot (Q_2)_{\alpha\beta,ij} \rangle_{L_x^2}}_{\mathcal{B}}. \end{aligned}$$

First, we control  $\mathcal{A}$  using a standard estimate:

$$\mathcal{A} \lesssim \|\nabla \delta Q\|_{L_x^2} \|\nabla \delta v\|_{L_x^2} \|\nabla Q_2\|_{L_x^\infty} \lesssim \|\nabla Q_2\|_{H^s}^2 \|\nabla \delta Q\|_{L_x^2}^2 + c_{\beta_4} \|\nabla \delta v\|_{L_x^2}^2.$$

The term  $\mathcal{B}$  requires a deeper analysis. First, we define the parameter  $\theta$  in  $(0, 1/2]$  as the minimum between  $1/2$  and  $s - d/2$ . Thus, since  $\Delta Q_2$  belongs to  $L^2(\mathbb{R}_+, H^{s-1}(\mathbb{R}^d))$ , then it belongs also to  $L^2(\mathbb{R}_+, H^{\theta+d/2-1}(\mathbb{R}^d))$ . Then we will make use of the following Sobolev embeddings:

$$\begin{aligned} H^{s-1}(\mathbb{R}^d) &\hookrightarrow H^{\theta+d/2-1}(\mathbb{R}^d) \hookrightarrow L^{\frac{d}{1-\theta}}(\mathbb{R}^d), \\ H^1(\mathbb{R}^d) &\hookrightarrow L^2(\mathbb{R}^d) \cap L^{\frac{2d}{d-2}}(\mathbb{R}^d) \hookrightarrow L^{\frac{2d}{d-2(1-\theta)}}(\mathbb{R}^d). \end{aligned} \tag{3.18}$$

Then  $\mathcal{B}$  is bounded by

$$\begin{aligned} \mathcal{B} &\lesssim \|\nabla \delta Q\|_{L_x^2} \|\delta v\|_{L_x^{\frac{2d}{d-2(1-\theta)}}} \|\Delta Q_2\|_{L_x^{\frac{d}{1-\theta}}} \lesssim \|\nabla \delta Q\|_{L_x^2} \|\delta v\|_{H^1} \|\Delta Q_2\|_{H^{\theta+\frac{d}{2}-1}} \\ &\lesssim \|\nabla \delta Q\|_{L_x^2} \|\delta v\|_{L^2} \|\Delta Q_2\|_{H^{s-1}} + \|\delta Q\|_{L_x^2} \|\nabla \delta v\|_{L^2} \|\Delta Q_2\|_{H^{s-1}} \\ &\lesssim \|\nabla Q_2\|_{H^s}^2 (\|\nabla \delta Q\|_{L_x^2}^2 + \|\delta v\|_{L_x^2}^2) + c_{\beta_4} \|\nabla \delta v\|_{L_x^2}^2 + c \|\nabla \delta Q\|_{L_x^2}^2. \end{aligned}$$

Summarizing, the second term is estimated as follows:

$$\langle \Delta \delta Q, \delta v \cdot \nabla Q_2 \rangle_{L_x^2} \lesssim \|\nabla Q_2\|_{H^s}^2 (\|\nabla \delta Q\|_{L_x^2}^2 + \|\delta v\|_{L_x^2}^2) + c_{\beta_4} \|\nabla \delta v\|_{L_x^2}^2 + c \|\nabla \delta Q\|_{L_x^2}^2.$$

Now, let us observe that  $\langle u_1 \cdot \nabla \delta \dot{Q}, \delta \dot{Q} \rangle_{L_x^2} = 0$  because of the free divergence condition of  $u_1$ . Moreover, always recalling the embeddings (3.18), we have

$$\begin{aligned} \langle \delta v \cdot \nabla \dot{Q}_2, \delta \dot{Q} \rangle_{L_x^2} &\lesssim \|\delta v\|_{L_x^{\frac{2d}{d-2(1-\theta)}}} \|\nabla \dot{Q}_2\|_{L_x^{\frac{d}{1-\theta}}} \|\delta Q\|_{L_x^2} \lesssim \|\delta v\|_{H^1} \|\nabla \dot{Q}_2\|_{H^{s-1}} \|\delta Q\|_{L_x^2} \\ &\lesssim \|\dot{Q}_2\|_{H^s} \|\delta v\|_{L^2} \|\delta Q\|_{L_x^2} + \|\dot{Q}_2\|_{H^s} \|\nabla \delta v\|_{L^2} \|\delta Q\|_{L_x^2} \\ &\lesssim \|\dot{Q}_2\|_{H^s}^2 (\|\delta v\|_{L_x^2}^2 + \|\delta Q\|_{L_x^2}^2) + c_{\beta_x} \|\nabla \delta v\|_{L_x^2}^2 + c_{\mu_1} \|\delta \dot{Q}\|_{L_x^2}^2. \end{aligned}$$

The remaining terms can easily be controlled by the Hölder inequality and the Sobolev embedding  $H^s \hookrightarrow L_x^\infty$ . First the terms related to the parameter  $a$  fulfill

$$\begin{aligned} \langle \delta Q, v_1 \cdot \nabla \delta Q \rangle_{L_x^2} &\lesssim \|\delta Q\|_{L_x^2} \|v_1\|_{L^\infty} \|\nabla \delta Q\|_{L_x^2} \lesssim \|v_1\|_{H^s} (\|\delta Q\|_{L_x^2}^2 + \|\nabla \delta Q\|_{L_x^2}^2), \\ \langle \delta Q, \delta v \cdot \nabla Q_2 \rangle_{L_x^2} &\lesssim \|\delta Q\|_{L_x^2} \|\delta v\|_{L_x^2} \|\nabla Q_2\|_{L_x^\infty} \lesssim \|\nabla Q_2\|_{H^s} (\|\delta Q\|_{L_x^2}^2 + \|\delta v\|_{L_x^2}^2), \end{aligned}$$

The terms related to  $b$  can be bounded as follows

$$\begin{aligned} \langle Q_1 \delta Q + \delta Q Q_2, \delta \dot{Q} \rangle_{L_x^2} &\lesssim \|(Q_1, Q_2)\|_{L_x^\infty} \|\delta Q\|_{L_x^2} \|\delta \dot{Q}\|_{L_x^2} \\ &\lesssim \|(Q_1, Q_2)\|_{H^s}^2 \|\delta Q\|_{L_x^2}^2 + c_{\mu_1} \|\delta \dot{Q}\|_{L_x^2}^2 \end{aligned}$$



and finally the one multiplied by  $c$  is estimated by

$$\langle \delta Q \operatorname{tr}\{Q_1^2\} + Q_2 \operatorname{tr}\{\delta Q Q_2\} + Q_2 \operatorname{tr}\{Q_2 \delta Q\}, \delta \dot{Q} \rangle_{L_x^2} \lesssim \|(Q_1, Q_2)\|_{H^s}^2 (\|\delta Q\|_{L_x^2}^2 + \|\delta \dot{Q}\|_{L_x^2}^2).$$

It remains to control the terms related to  $\mu_1$  and  $\mu_2$  which can be handled through

$$\langle \delta A, \delta \dot{Q} \rangle_{L_x^2} \lesssim \|\delta A\|_{L_x^2} \|\delta \dot{Q}\|_{L_x^2} \lesssim \|\delta \dot{Q}\|_{L_x^2}^2 + c_{\beta_4} \|\nabla \delta v\|_{L_x^2}^2$$

and

$$\begin{aligned} \langle [\Omega_1, \delta Q] + [\delta \Omega, Q_2], \delta \dot{Q} \rangle_{L_x^2} &\lesssim (\|\nabla v_1\|_{H^s}^2 + \|Q_2\|_{H^s}^2) (\|\delta Q\|_{L_x^2}^2 + \|\delta \dot{Q}\|_{L_x^2}^2) + \\ &\quad + c_{\mu_1} \|\delta \dot{Q}\|_{L_x^2}^2 + c_{\beta_4} \|\nabla \delta v\|_{L_x^2}^2. \end{aligned}$$

Using all the previous estimates in the equality (3.17), we obtain

$$\begin{aligned} \frac{d}{dt} \left[ \frac{J}{2} \|\delta \dot{Q}\|_{L_x^2}^2 + \frac{L}{2} \|\nabla \delta Q\|_{L_x^2}^2 + \frac{a}{2} \|\delta Q\|_{L_x^2}^2 \right] + \mu_1 \|\delta \dot{Q}\|_{L_x^2}^2 &\lesssim \left( 1 + \|Q_2\|_{H^s}^2 + \|\nabla v_1\|_{H^s}^2 + \|v_1\|_{H^s}^2 + \right. \\ &\quad \left. + \|\dot{Q}_2\|_{H^s}^2 + \|\nabla Q_2\|_{H^s}^2 + \|Q_1\|_{H^s}^2 \right) \left( \|\delta v\|_{L_x^2}^2 + \|\delta \dot{Q}\|_{L_x^2}^2 + \|\delta Q\|_{L_x^2}^2 + \|\nabla \delta Q\|_{L_x^2}^2 \right) + \\ &\quad + c_{\beta_4} \|\nabla \delta v\|_{L_x^2}^2 + c_{\mu_1} \|\delta \dot{Q}\|_{L_x^2}^2. \end{aligned} \quad (3.19)$$

Now let us take in consideration the difference between the momentum equations of the two solutions, namely

$$\begin{aligned} \partial_t \delta v + v_1 \cdot \nabla \delta v + \delta v \cdot \nabla v_2 - \frac{\beta_4}{2} \Delta \delta v &= -L \nabla \cdot \left\{ \nabla \delta Q \otimes \nabla Q_1 + \nabla Q_2 \otimes \nabla \delta Q \right\} - \\ + \beta_1 \nabla \cdot \left\{ \operatorname{tr}\{\delta Q A_1\} Q_1 + \operatorname{tr}\{Q_2 \delta A\} Q_1 + \operatorname{tr}\{Q_2 A_2\} \delta Q \right\} &+ \beta_5 \nabla \cdot \{A_1 \delta Q + \delta A Q_2\} + \\ + \beta_6 \nabla \cdot \{\delta Q A_1 + Q_2 \delta A\} &+ \frac{\mu_2}{2} \nabla \cdot \left\{ \delta \dot{Q} - [\delta \Omega, Q_1] - [\Omega_2, \delta Q] \right\} + \\ + \mu_1 \nabla \cdot \left\{ [\delta Q, (\dot{Q}_1 - [\Omega_1, Q_1])] + [Q_2, (\delta \dot{Q} - [\delta \Omega, Q_1] - [\Omega_2, \delta Q])] \right\}. \end{aligned} \quad (3.20)$$

We proceed similarly as before, multiplying both the left and right-hand sides by  $\delta v$  and integrating everything over  $\mathbb{R}^d$ , and by parts, obtaining

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\delta v\|_{L_x^2}^2 + \frac{\beta_4}{2} \|\nabla \delta v\|_{L_x^2}^2 &= L \langle \nabla \delta Q \otimes \nabla Q_1 + \nabla Q_2 \otimes \nabla \delta Q, \nabla \delta v \rangle_{L_x^2} + \\ - \beta_1 \langle \operatorname{tr}\{\delta Q A_1\} Q_1 + \operatorname{tr}\{Q_2 A_2\} \delta Q, \nabla \delta v \rangle_{L_x^2} &+ \beta_1 \langle \operatorname{tr}\{Q_2 \delta A\} Q_1, \nabla \delta v \rangle_{L_x^2} - \\ - \beta_5 \langle A_1 \delta Q + \delta A Q_2, \nabla \delta v \rangle_{L_x^2} - \beta_6 \langle \delta Q A_1 + Q_2 \delta A, \nabla \delta v \rangle_{L_x^2} &- \frac{\mu_2}{2} \langle \delta \dot{Q}, \nabla \delta v \rangle_{L_x^2} + \\ + \frac{\mu_2}{2} \langle [\delta \Omega, Q_1] + [\Omega_2, \delta Q], \nabla \delta v \rangle_{L_x^2} - \mu_1 \langle [\delta Q, \dot{Q}_1], \nabla \delta v \rangle_{L_x^2} &- \mu_1 \langle [Q_2, \delta \dot{Q}], \nabla \delta v \rangle_{L_x^2} - \\ + \mu_1 \langle [Q_2, [\delta \Omega, Q_1] + [\Omega_2, \delta Q]], \nabla \delta v \rangle_{L_x^2} &+ \mu_1 \langle [\delta Q, [\Omega_1, Q_1]], \nabla \delta v \rangle_{L_x^2} \\ - \langle v_1 \cdot \nabla \delta v, \delta v \rangle_{L_x^2} - \langle \delta v \cdot \nabla v_2, \delta v \rangle_{L_x^2}, \end{aligned} \quad (3.21)$$

then we proceed estimating each term on the right-hand side. At first, we have

$$\begin{aligned} \langle \nabla \delta Q \otimes \nabla Q_1 + \nabla Q_2 \otimes \nabla \delta Q, \nabla \delta v \rangle_{L_x^2} &\lesssim \left( \|\nabla Q_1\|_{L_x^\infty} + \|\nabla Q_2\|_{L_x^\infty} \right) \|\nabla \delta Q\|_{L_x^2} \|\nabla \delta v\|_{L_x^2} \\ &\lesssim \left( \|\nabla Q_1\|_{H^s}^2 + \|\nabla Q_2\|_{H^s}^2 \right) \|\nabla \delta Q\|_{L_x^2}^2 + c_{\beta_4} \|\nabla \delta v\|_{L_x^2}^2, \end{aligned}$$

while the terms concerning  $\beta_1$  are handled by

$$\begin{aligned} \langle \text{tr}\{\delta Q A_1\} Q_1 + \text{tr}\{Q_2 A_2\} \delta Q, \nabla \delta v \rangle_{L_x^2} &\lesssim \\ &\lesssim \left( \|\nabla u_1\|_{L_x^\infty} \|Q_1\|_{L_x^\infty} + \|\nabla u_2\|_{L_x^\infty} \|Q_2\|_{L_x^\infty} \right) \|\delta Q\|_{L_x^2} \|\nabla \delta v\|_{L_x^2} \\ &\lesssim \left( \|\nabla v_1\|_{H^s}^2 \|Q_1\|_{H^s}^2 + \|\nabla v_2\|_{H^s}^2 \|Q_2\|_{H^s}^2 \right) \|\delta Q\|_{L_x^2}^2 + c_{\beta_4} \|\nabla \delta v\|_{L_x^2}^2, \end{aligned}$$

and

$$\langle \text{tr}\{Q_2 \delta A\} Q_1, \nabla \delta v \rangle_{L_x^2} \lesssim \|Q_1\|_{L_x^\infty} \|Q_2\|_{L_x^\infty} \|\nabla \delta v\|_{L_x^2}^2 \lesssim \|Q_1\|_{H^s} \|Q_2\|_{H^s} \|\nabla \delta v\|_{L_x^2}^2.$$

Now, we bound the terms related to  $\beta_5$  and  $\beta_6$  as follows:

$$\begin{aligned} \langle A_1 \delta Q + \delta A Q_2, \nabla \delta v \rangle_{L_x^2} &\lesssim \|\nabla v_1\|_{L^\infty} \|\delta Q\|_{L_x^2} \|\nabla \delta v\|_{L_x^2} + \|Q_2\|_{L^\infty} \|\nabla \delta v\|_{L_x^2}^2 \\ &\lesssim \|\nabla v_1\|_{H^s}^2 \|\delta Q\|_{L_x^2}^2 + (c_{\beta_4} + \|Q_2\|_{H^s}) \|\nabla \delta v\|_{L_x^2}^2, \\ \langle \delta Q A_1 + Q_2 \delta A, \nabla \delta v \rangle_{L_x^2} &\lesssim \|\delta Q\|_{L_x^2} \|\nabla v_1\|_{L_x^\infty} \|\nabla \delta v\|_{L_x^2} + \|Q_2\|_{L_x^\infty} \|\nabla \delta v\|_{L_x^2}^2 \\ &\lesssim \|\nabla v_1\|_{H^s}^2 \|\delta Q\|_{L_x^2}^2 + (c_{\beta_4} + \|Q_2\|_{H^s}) \|\nabla \delta v\|_{L_x^2}^2. \end{aligned}$$

Now, we move on and bound the terms related to  $\mu_2$  by

$$\langle \delta \dot{Q}, \nabla \delta v \rangle_{L_x^2} \lesssim \|\delta \dot{Q}\|_{L_x^2}^2 + c_{\beta_4} \|\nabla \delta v\|_{L_x^2}^2,$$

$$\begin{aligned} \langle [\delta \Omega, Q_1] + [\Omega_2, \delta Q], \nabla \delta v \rangle_{L_x^2} &\lesssim \|Q_1\|_{L_x^\infty} \|\nabla \delta v\|_{L_x^2}^2 + \|\nabla v_2\|_{L_x^\infty} \|\delta Q\|_{L_x^2} \|\nabla \delta v\|_{L_x^2} \\ &\lesssim \|\nabla v_2\|_{H_x^s}^2 \|\delta Q\|_{L_x^2}^2 + (c_{\beta_4} + \|Q_1\|_{H_x^s}) \|\nabla \delta v\|_{L_x^2}^2, \end{aligned}$$

while the terms related to  $\mu_1$  can be handled by

$$\begin{aligned} \langle [\delta Q, \dot{Q}_1], \nabla \delta v \rangle_{L_x^2} &\lesssim \|\delta Q\|_{L_x^2} \|\dot{Q}_1\|_{L_x^\infty} \|\nabla \delta v\|_{L_x^2} \lesssim \|\dot{Q}_1\|_{H^s}^2 \|\delta Q\|_{L_x^2}^2 + c_{\beta_4} \|\nabla \delta v\|_{L_x^2}^2, \\ \langle [Q_2, \delta \dot{Q}], \nabla \delta v \rangle_{L_x^2} &\lesssim \|Q_2\|_{L_x^\infty} \|\delta \dot{Q}\|_{L_x^2} \|\nabla \delta v\|_{L_x^2} \lesssim \|Q_2\|_{H^s}^2 \|\delta \dot{Q}\|_{L_x^2}^2 + c_{\beta_4} \|\nabla \delta v\|_{L_x^2}^2 \end{aligned}$$

and also

$$\begin{aligned} \langle [Q_2, [\delta \Omega, Q_1] + [\Omega_2, \delta Q]], \nabla \delta v \rangle_{L_x^2} &\lesssim \|Q_2\|_{L_x^\infty} \|Q_1\|_{L_x^\infty} \|\nabla \delta v\|_{L_x^2}^2 + \|Q_2\|_{L_x^\infty} \|\nabla v_2\|_{L_x^\infty} \|\delta Q\|_{L_x^2} \|\nabla \delta v\|_{L_x^2} \\ &\lesssim \|Q_2\|_{H_x^s}^2 \|\nabla v_2\|_{H_x^s}^2 \|\delta Q\|_{L_x^2}^2 + \left( c_{\beta_4} + \|Q_2\|_{H_x^s} \|Q_1\|_{H_x^s} \right) \|\nabla \delta v\|_{L_x^2}^2, \end{aligned}$$

$$\begin{aligned} \langle [\delta Q, [\Omega_1, Q_1]], \nabla \delta v \rangle_{L_x^2} &\leq \|Q_1\|_{L^\infty} \|\nabla v_1\|_{L^\infty} \|\delta Q\|_{L_x^2} \|\nabla \delta v\|_{L_x^2} \\ &\leq \|Q_1\|_{H^s}^2 \|\nabla v_1\|_{H^s}^2 \|\delta Q\|_{L^2}^2 + c_{\beta_4} \|\nabla \delta v\|_{L^2}^2 \end{aligned}$$

Finally, let us remark that  $\langle v_1 \cdot \nabla \delta v, \delta v \rangle_{L_x^2} = 0$  and

$$\langle \delta v \cdot \nabla v_2, \delta v \rangle_{L_x^2} \lesssim \|\nabla v_2\|_{L_x^\infty} \|\delta v\|_{L_x^2}^2 \lesssim \|\nabla v_2\|_{H^s} \|\delta v\|_{L_x^2}^2.$$

Thus, summarizing all the previous estimates and using them in (3.20) we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\delta v\|_{L_x^2}^2 + \frac{\beta_4}{2} \|\nabla \delta v\|_{L_x^2}^2 &\lesssim \left\{ 1 + \|\nabla v_2\|_{H^s} + \|\nabla v_1\|_{H^s}^2 + \|\nabla v_2\|_{H^s}^2 + \|\nabla Q_1\|_{H^s}^2 \right. \\ &\quad \left. + \|Q_2\|_{H^s}^2 + \|\nabla Q_2\|_{H^s}^2 + \|\nabla v_1\|_{H^s}^2 \|Q_1\|_{H^s}^2 + \|\nabla v_2\|_{H^s}^2 \|Q_2\|_{H^s}^2 + \|\dot{Q}_1\|_{H^s}^2 \right\} \\ &\quad \times \left\{ \|\delta v\|_{L_x^2}^2 + \|\nabla \delta Q\|_{L_x^2}^2 + \|\delta Q\|_{L_x^2}^2 + \|\delta \dot{Q}\|_{L_x^2}^2 \right\} + \\ &\quad + \left\{ c_{\beta_4} + \|Q_2\|_{H_x^s} \|Q_1\|_{H_x^s} + \|Q_1\|_{H_x^s} + \|Q_2\|_{H_x^s} \right\} \|\nabla \delta v\|_{L_x^2}^2 \end{aligned} \quad (3.22)$$

Now, defining the functions  $\Psi = \Psi(t)$  and  $f = f(t)$  by

$$\begin{aligned} \Psi &:= \frac{1}{2} \|\delta v\|_{L_x^2}^2 + \frac{J}{2} \|\delta \dot{Q}\|_{L_x^2}^2 + \frac{L}{2} \|\nabla \delta Q\|_{L_x^2}^2 + \frac{a}{2} \|\delta Q\|_{L_x^2}^2 \\ f &:= \left\{ 1 + \|Q_1\|_{H^s}^2 \|\nabla v_2\|_{H^s} + \|\nabla v_1\|_{H^s}^2 + \|\nabla v_2\|_{H^s}^2 + \|\nabla Q_1\|_{H^s}^2 + \|Q_2\|_{H^s}^2 + \right. \\ &\quad \left. + \|\nabla Q_2\|_{H^s}^2 + \|\nabla v_1\|_{H^s}^2 \|Q_1\|_{H^s}^2 + \|\nabla v_2\|_{H^s}^2 \|Q_2\|_{H^s}^2 + \|\dot{Q}_1\|_{H^s}^2 \right\}, \end{aligned}$$

and observing that  $f \in L_{loc}^1(\mathbb{R}_+)$ , we finally take the sum between (3.19) and (3.22), achieving

$$\begin{aligned} \frac{d}{dt} \Psi + \mu_1 \|\delta \dot{Q}\|_{L_x^2}^2 + \frac{\beta_4}{2} \|\nabla \delta v\|_{L_x^2}^2 &\lesssim f \Psi + c_{\mu_1} \|\delta \dot{Q}\|_{L_x^2}^2 + \\ &\quad + \left\{ c_{\beta_4} + \|Q_2\|_{H_x^s} \|Q_1\|_{H_x^s} + \|Q_1\|_{H_x^s} + \|Q_2\|_{H_x^s} \right\} \|\nabla \delta v\|_{L_x^2}^2. \end{aligned}$$

Hence, assuming  $c_{\beta_4}$ ,  $c_{\mu_1}$  and the initial data small enough, we can absorb by the left-hand side the terms related to  $\|\delta \dot{Q}\|_{L_x^2}^2$  and  $\|\nabla \delta v\|_{L_x^2}^2$  on the right-hand side, so that the following inequality is fulfilled:

$$\frac{d}{dt} \Psi \lesssim f \Psi.$$

Then, since  $\Psi(0) = 0$ , the Gronwall's inequality yields  $\Psi$  to be constantly null, especially

$$\delta v = v_1 - v_2 = 0 \quad \text{and} \quad \delta Q = Q_1 - Q_2 = 0. \quad (3.23)$$

This concludes the proof of Theorem 1.2.  $\square$

#### 4. Twist waves

In the following we consider an example of a “twist-wave” solution, that is a solution of the coupled system for which the flow  $v$  is zero. As noted in the introduction, this amounts to determining a solution of the  $Q$ -tensor equation (1.3) with zero flow (hence  $v = \Omega = A = 0$ ,  $\dot{Q}$  becomes  $\partial_t Q$ ) but satisfying *an additional nonlinear constraint* namely (1.21).

Our ansatz is inspired from stationary-case studies [3], namely *the “melting-hedgehog wave”* obtained by taking in  $\mathbb{R}^d$  with  $d = 2$  or  $d = 3$  the ansatz:

$$T(t, x) := f(t, |x|) \bar{H}(x)$$

with  $f : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$  a function to be determined and  $\bar{H}$  the “hedgehog” function (see [3] for details about its physical significance) :

$$\bar{H}_{ij}(x) := \frac{x_i x_j}{|x|^2} - \frac{\delta_{ij}}{d}, i, j = 1, \dots, d$$

Let us note that in order to avoid a discontinuity at 0 for  $T$  we need to take

$$f(t, 0) = 0, \forall t \geq 0 \quad (4.1)$$

i.e. the hedgehog “melts” at the origin.

Then one can check that the equation (1.20) reduces to an equation for  $f$  only, namely:

$$Jf_{tt} + \mu_1 f_t = L \left( f_{rr} + f_r \frac{d-1}{r} - \frac{2d}{r^2} f \right) - af + \frac{b(d-2)}{d} f^2 - \frac{c(d-1)}{d} f^3 \quad (4.2)$$

Note that in order to avoid a singularity at the origin for  $f$  we need to further have that

$$f_r(t, 0) = 0, \forall t \geq 0 \quad (4.3)$$

Condition (4.3) is an apparent singularity, having to do with the radial symmetry, because if one denotes  $G(t, x) = f(t, |x|)$  then (4.3) holds for  $G$  sufficiently smooth in the  $x$  variable.

On the other hand in order to check that the constraint equation (1.21) holds it suffices to check that both  $\nabla \cdot T_t$  and  $\nabla \cdot (\nabla T \otimes \nabla T)$  can be expressed as gradients <sup>8</sup>.

We have:

$$\nabla \cdot T_t = \left( f_t(t, |x|) \left( \frac{x_i x_j}{|x|^2} - \frac{\delta_{ij}}{d} \right) \right)_{,j} = (d-1) \left( \frac{1}{d} \partial_r f_t(t, r) + \frac{f_t(t, r)}{r} \right) \frac{x_i}{|x|} \quad (4.4)$$

Thus, in order to have the function in  $r$  bounded up to 0 we will need to assume a similar condition as in (4.3) but for the time derivative:

$$f_{tr}(t, 0) = 0, \forall t \geq 0 \quad (4.5)$$

---

<sup>8</sup>since the commutator appearing in (1.21) vanishes for our ansatz

Under this assumption have that for sufficiently smooth  $f$  there exists a function  $g : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$  such that  $g_r = (d-1) \left( \frac{1}{d} \frac{\partial^2 f}{\partial r \partial t}(t, r) + \frac{f_t(t, r)}{r} \right)$ , hence  $\nabla \cdot T_t = \nabla_x g(t, |x|)$ .

Furthermore, we have:

$$\nabla \cdot (T \otimes T) = (T_{kl,i} T_{kl,j})_{,j} = T_{kl,ij} T_{kl,i} + T_{kl,i} \Delta T_{kl}$$

As  $T_{kl,ij} T_{kl,j} = \frac{1}{2} (|\nabla T|)_{,i}$  it suffices to check that  $T_{kl,i} \Delta T_{kl}$  is a gradient, which in our case amounts to checking that

$$f_r(f_{rr} + \frac{(d-1)f_r}{r} - \frac{2df}{r^2}) \frac{x_i}{|x|} \left( \frac{x_k x_l}{|x|^2} - \frac{\delta_{kl}}{d} \right) \left( \frac{x_k x_l}{|x|^2} - \frac{\delta_{kl}}{d} \right) = \frac{d-1}{d} f_r(f_{rr} + \frac{(d-1)f_r}{r} - \frac{2df}{r^2}) \frac{x_i}{|x|}$$

is a gradient.

Thus, assuming (4.1) and (4.3) we have that there exists  $h : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$  such that  $h_r = \frac{d-1}{d} f_r(f_{rr} + \frac{(d-1)f_r}{r} - \frac{2df}{r^2})$ , hence  $\nabla_x h(t, |x|) = \nabla \cdot (\nabla T \otimes \nabla T)$ .

These formal computations provide indeed a twist wave under the smoothness conditions mentioned before. In order to make the above rigorous we continue with the proof of Proposition 1.3:

**Proof.** We proceed in several steps. We will first show that the  $Q$ -tensor equation (1.20) has global in time strong solutions for arbitrary initial data. We then prove a weak-strong uniqueness result for solutions of the  $Q$ -tensor equation (1.20) and show that the  $f$ -equation (1.22) has weak solutions and then that these weak solutions provide a global weak solution of (1.20). We use the weak-strong uniqueness to conclude that the solutions provided by  $f(t, |x|) \bar{H}(x)$  are twist-wave solutions.

### Step 1: Global strong solutions of (1.20)

We just provide here the apriori estimates necessary for obtaining the weak and strong solutions. The actual construction through an approximation scheme can be done similarly as in the proof of Theorem 1.2.

We first obtain the apriori boundedness of the  $L^2$  norm. To this end we multiply (1.20) by  $T_t$ , integrate over  $\mathbb{R}^d$  and by parts<sup>9</sup>, to get:

$$\frac{d}{dt} \int_{\mathbb{R}^d} \frac{J}{2} |T_t|^2 + \frac{L}{2} |\nabla T|^2 + \psi_B(T) dx + \mu_1 \int_{\mathbb{R}^d} |T_t|^2 dx = 0 \quad (4.6)$$

We note that because  $\psi_B(T)$  can be negative this does not suffice for obtaining estimates on the  $L^2$  norm of  $T$ . Thus we multiply (1.20) by  $T$ , integrate over  $\mathbb{R}^d$  and by parts, to get:

---

<sup>9</sup> throughout the proof we always assume as usually that we can integrate by parts without boundary terms

$$\begin{aligned} \frac{J}{2} \frac{d^2}{dt^2} \int_{\mathbb{R}^d} |T|^2 dx - J \int_{\mathbb{R}^d} |T_t|^2 dx + \frac{\mu_1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} |T|^2 dx + L \int_{\mathbb{R}^d} |\nabla T|^2 dx = \\ \int_{\mathbb{R}^d} (-a|T|^2 + b \operatorname{tr}(T^3) - c|T|^4) dx \end{aligned} \quad (4.7)$$

We multiply (4.6) by  $J$  and (4.7) by  $\mu_1$  and add them together to get:

$$\begin{aligned} \frac{J\mu_1}{2} \frac{d^2}{dt^2} \int_{\mathbb{R}^d} |T|^2 dx + \frac{d}{dt} \int_{\mathbb{R}^d} \frac{J^2}{2} |T_t|^2 + \frac{LJ}{2} |\nabla T|^2 + J\psi_B(T) + \frac{\mu_1^2}{2} |T|^2 dx \\ + L\mu_1 \int_{\mathbb{R}^d} |\nabla T|^2 dx \leq C \int_{\mathbb{R}^d} |T|^2 dx \end{aligned} \quad (4.8)$$

with the large enough constant  $C$  depending just on  $\mu_1$ ,  $a$ ,  $b$  and  $c$ .

Integrating over  $[0, t]$  we obtain:

$$\begin{aligned} \frac{J\mu_1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} |T|^2(t, x) dx + \int_{\mathbb{R}^d} \left( \frac{J^2}{2} |T_t|^2 + \frac{LJ}{2} |\nabla T|^2 + J\psi_B(T) + \frac{\mu_1^2}{2} |T|^2 \right) (t, x) dx \leq \\ \leq J\mu_1 \int_{\mathbb{R}^d} (T_t : T)(0, x) dx + \int_{\mathbb{R}^d} \left( \frac{J^2}{2} |T_t|^2 + \frac{LJ}{2} |\nabla T|^2 + J\psi_B(T) + \frac{\mu_1^2}{2} |T|^2 \right) (0, x) dx \\ + C \int_0^t \int_{\mathbb{R}^d} |T|^2(s, x) ds dx \end{aligned} \quad (4.9)$$

which implies:

$$\frac{J\mu_1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} |T|^2(t, x) dx + \int_{\mathbb{R}^d} \frac{J^2}{2} |T_t|^2(t, x) dx \leq C + C \int_{\mathbb{R}^d} |T(t, x)|^2 dx + \int_0^t \int_{\mathbb{R}^d} |T|^2(s, x) ds dx \quad (4.10)$$

with the constants  $C$  depending only on the initial data and the constants in the equation. Thus using Gronwall inequality and Fubini (to turn the double time integral into a weighted time integral) together with (4.6) we obtain apriori control over certain energy-level norms:

$$T \in L^\infty(0, T; L^2) \cap L^\infty(0, T; H^1) \cap L^\infty(0, T; L^4) < C(a, b, c, J, \mu_1, T, \|T_0\|_{H^1}, \|\partial_t T_0\|_{L^2}) \quad (4.11)$$

$$T_t \in L^\infty(0, T; L^2) < C(a, b, c, J, \mu_1, T, \|T_0\|_{L^2}, \|\partial_t T_0\|_{L^2}) \quad (4.12)$$

In order to obtain control over the  $H^s$  norm we multiply (1.20) with  $T_t$  in the  $H^s$  inner product, with  $s > \frac{d}{2}$  obtaining:

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} (J \|T_t\|_{H^s}^2 + L \|\nabla T\|_{H^s}^2) + \mu_1 \|T_t\|_{H^s}^2 dx &= \langle T_t, -aT + b(T^2 - \frac{|T|^2}{d} I_d) - cT|T|^2 \rangle_{H^s} \\
&\leq C \|T_t\|_{H^s} (\|T\|_{H^s} + \|T\|_{H^s}^3) \\
&\leq C \|T_t\|_{H^s}^2 + C \|T\|_{H^s}^2 + \varepsilon \|T\|_{H^s}^6 \\
&\leq C \|T_t\|_{H^s}^2 + C \|T\|_{H^s}^2 + \varepsilon \|T\|_{H^1}^{\frac{6}{s}} \cdot \varepsilon \|T\|_{H^{s+1}}^{\frac{6(s-1)}{s}}
\end{aligned} \tag{4.13}$$

where in the last inequality we estimated  $H^s$  through interpolation between  $H^1$  and  $H^{s+1}$ . In order to be able to control the term  $\varepsilon \|T\|_{H^s}^{\frac{6(s-1)}{s}}$  we need<sup>10</sup> to have  $\frac{6(s-1)}{s} \leq 2$  i.e.  $s \leq \frac{3}{2}$ . Thus using Gronwal the previous lemma gives apriori control on  $\|T_t\|_{H^s}^2 + \|T\|_{H^{s+1}}^2$  in  $L^\infty(0, T)$  for any  $T > 0$ , provided that  $s \leq \frac{3}{2}$ . We can then repeat the same estimates as above but at the last line estimate through interpolation of  $H^s$  between  $H^{\frac{3}{2}}$  and  $H^{s+1}$  with  $s \leq \frac{5}{2}$ . Repeating inductively we obtain control for arbitrary  $s > \frac{d}{2}$ .

## Step 2: Weak-strong uniqueness

We assume that the weak solutions have regularity:

$$T \in L^\infty(0, T; L^2) \cap L^\infty(0, T; H^1) \cap L^\infty(0, T; L^4) \tag{4.14}$$

$$T_t \in L^\infty(0, T; L^2) \tag{4.15}$$

We consider the difference of two solutions  $T_1$  and  $T_2$  of (1.20) with  $T_1$  being a weak solution and  $T_2$  a strong solution. We denote  $\delta T := T_1 - T_2$ , and note that

$$\delta T(0, x) = \partial_t \delta T(0, x) \equiv 0$$

and  $\delta T$  satisfies the equation:

$$\begin{aligned}
J \delta T_{tt} + \mu_1 \delta T_t &= L \Delta \delta T - a \delta T + b \left( \delta T^2 + T_2 \delta T + \delta T T_2 - \frac{|\delta T|^2}{d} I_d - \frac{2T_2 : \delta T}{d} I_d \right) \\
&\quad - c [\delta T |\delta T|^2 + \delta T |T_2|^2 + 2\delta T (\delta T : T_2) + T_2 |\delta T|^2 + 2T_2 (\delta T : T_2)]
\end{aligned} \tag{4.16}$$

We multiply the last relation by  $\delta T_t$ , integrate over  $\mathbb{R}^d$  and by parts, to get:

---

<sup>10</sup> In here we need  $d = 2$  as we assumed that  $\frac{d}{2} < s$  and  $d = 3$  would contradict the restriction  $s \leq \frac{3}{2}$

$$\begin{aligned}
& \frac{d}{dt} \int_{\mathbb{R}^d} \frac{J}{2} |\delta T_t|^2 + \frac{L}{2} |\nabla \delta T|^2 + \psi_B(\delta T) dx + \mu_1 \int_{\mathbb{R}^d} |\delta T_t|^2 dx = b \int_{\mathbb{R}^d} (T_2 \delta T + \delta T T_2) : \delta T_t dx \\
& - c \int_{\mathbb{R}^d} |T_2|^2 (\delta T : \delta T_t) + 2(\delta T : T_2)(\delta T : \delta T_t) + |\delta T|^2 (T_2 : \delta T_t) + 2(\delta T : T_2)(\delta T_t : T_2) dx \\
& \leq C \left( \int_{\mathbb{R}^d} |\delta T|^2 + |\delta T|^4 dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^d} |\delta T_t|^2 dx \right)^{\frac{1}{2}}
\end{aligned} \tag{4.17}$$

with  $C$  a constant depending on  $\|T_2\|_{L^\infty}$ ,  $b$  and  $c$ .

On the other hand, multiplying (4.16) by  $\delta T$ , integrating over  $\mathbb{R}^d$  and by parts, we get:

$$\begin{aligned}
& \frac{J}{2} \frac{d^2}{dt^2} \int_{\mathbb{R}^d} |\delta T|^2 dx - J \int_{\mathbb{R}^d} |\delta T_t|^2 dx + \frac{\mu_1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} |\delta T|^2 dx + L \int_{\mathbb{R}^d} |\nabla \delta T|^2 dx = \\
& \int_{\mathbb{R}^d} (-a |\delta T|^2 + b \operatorname{tr}(\delta T^3) - c |\delta T|^4) dx + b \int_{\mathbb{R}^d} (T_2 \delta T + \delta T T_2) : \delta T dx \\
& - c \int_{\mathbb{R}^d} (|\delta T|^2 |T_2|^2 + 3(\delta T : T_2) |\delta T|^2 + 2(\delta T : T_2)^2) dx \\
& \leq C \int_{\mathbb{R}^d} |\delta T|^2 - \frac{c}{2} \int_{\mathbb{R}^d} |\delta T|^4 dx
\end{aligned} \tag{4.18}$$

with  $C$  a constant depending on  $\|T_2\|_{L^\infty}$ ,  $a$ ,  $b$  and  $c$ .

We multiply (4.17) by  $J$  and (4.18) by  $\mu_1$  and add them together to get:

$$\begin{aligned}
& \frac{J\mu_1}{2} \frac{d^2}{dt^2} \int_{\mathbb{R}^d} |\delta T|^2 dx + \frac{d}{dt} \int_{\mathbb{R}^d} \frac{J^2}{2} |\delta T_t|^2 + \frac{LJ}{2} |\nabla \delta T|^2 + J\psi_B(\delta T) + \frac{\mu_1^2}{2} |\delta T|^2 dx + L\mu_1 \int_{\mathbb{R}^d} |\nabla \delta T|^2 dx \\
& \leq C \int_{\mathbb{R}^d} |\delta T_t|^2 + |\delta T|^2 dx
\end{aligned} \tag{4.19}$$

with  $C$  a constant depending on  $\|T_2\|_{L^\infty}$ ,  $a$ ,  $b$  and  $c$ .

Integrating over  $[0, t]$  we obtain:

$$\begin{aligned}
& \frac{J\mu_1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} |\delta T|^2(t, x) dx + \int_{\mathbb{R}^d} \left( \frac{J^2}{2} |\delta T_t|^2 + \frac{LJ}{2} |\nabla \delta T|^2 + J\psi_B(\delta T) + \frac{\mu_1^2}{2} |\delta T|^2 \right) (t, x) dx \leq \\
& \leq J\mu_1 \int_{\mathbb{R}^d} \underbrace{(\delta T_t : \delta T)(0, x)}_{=0} dx + \int_{\mathbb{R}^d} \underbrace{\left( \frac{J^2}{2} |\delta T_t|^2 + \frac{LJ}{2} |\nabla \delta T|^2 + J\psi_B(\delta T) + \frac{\mu_1^2}{2} |\delta T|^2 \right)}_{=0} (0, x) dx \\
& + C \int_0^t \int_{\mathbb{R}^d} (|\delta T_t|^2 + |\delta T|^2)(s, x) ds dx
\end{aligned} \tag{4.20}$$



which implies:

$$\begin{aligned} \frac{J\mu_1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} |\delta T|^2(t, x) dx + \int_{\mathbb{R}^d} \frac{J^2}{2} |\delta T_t|^2(t, x) dx \\ \leq C \int_{\mathbb{R}^d} |\delta T(t, x)|^2 dx + C \int_0^t \int_{\mathbb{R}^d} (|\delta T_t|^2 + |\delta T|^2)(s, x) ds dx \end{aligned} \quad (4.21)$$

Integrating one more time and using  $\delta T_t(0, \cdot) = \delta T(0, \cdot) \equiv 0$  and Gronwal we get that  $\delta T_t(t, \cdot) = \delta T(t, \cdot) \equiv 0$  for all  $t > 0$ .

### Step 3: weak solutions of the $f$ -equation (4.2)

We will just provide here the apriori estimates necessary for obtaining the weak solutions. The actual construction of the approximation scheme can be done through a straightforward modification of the scheme used in the proof of Theorem 1.2 and it is left to the interested reader.

We assume that (4.2) has a classical solution. We multiply it by  $f_t r^2$  and integrate over  $\mathbb{R}$  to get

$$\frac{d}{dt} \int_{\mathbb{R}} \left\{ \frac{J}{2} f_t^2 + \frac{L}{2} f_r^2 + h_B(f) + d \frac{f^2}{r^2} \right\} r^2 dr + \mu_1 \int_{\mathbb{R}} f_t^2 r^2 dr = 0 \quad (4.22)$$

where we used the reduced potential

$$h_B(f) = \frac{a}{2} f^2 - \frac{b(d-2)}{3d} f^3 + \frac{c(d-1)}{4d} f^4 \quad (4.23)$$

On the other hand, multiplying (4.2) by  $f r^2$  and integrating over  $\mathbb{R}$  we obtain:

$$\begin{aligned} \frac{d^2}{dt^2} \int_{\mathbb{R}} \frac{J}{2} f^2 r^2 dr - J \int_{\mathbb{R}} f_t^2 r^2 dr + \frac{\mu_1}{2} \frac{d}{dt} \int_{\mathbb{R}} f^2 r^2 dr + L \int_{\mathbb{R}} f_r^2 r^2 dr = \\ -2d \int_{\mathbb{R}} f^2 dr + \int_{\mathbb{R}} \left( -a f^2 + \frac{b(d-2)}{d} f^3 - \frac{c(d-1)}{d} f^4 \right) r^2 dr \end{aligned} \quad (4.24)$$

We now multiply (4.22) by  $J$  and add to it (4.24) multiplied by  $\mu_1$  to get:

$$\begin{aligned} \frac{d^2}{dt^2} \int_{\mathbb{R}} \frac{J\mu_1}{2} f^2 r^2 dr + \frac{d}{dt} \int_{\mathbb{R}} J \left\{ \frac{J}{2} f_t^2 + \frac{L}{2} f_r^2 + h_B(f) + d \frac{f^2}{r^2} \right\} r^2 dr + \frac{\mu_1^2}{2} \frac{d}{dt} \int_{\mathbb{R}} f^2 r^2 dr = \\ -L\mu_1 \int_{\mathbb{R}} f_r^2 r^2 dr - 2d\mu_1 \int_{\mathbb{R}} f^2 dr + \int_{\mathbb{R}} \mu_1 \left( -a f^2 + \frac{b(d-2)}{d} f^3 - \frac{c(d-1)}{d} f^4 \right) r^2 dr \end{aligned} \quad (4.25)$$

Integrating over  $[0, t]$  we get:

$$\begin{aligned}
& \frac{d}{dt} \int_{\mathbb{R}} \frac{J\mu_1}{2} f^2(t, r) r^2 dr + \int_{\mathbb{R}} J \left\{ \frac{J}{2} f_t^2 + \frac{L}{2} f_r^2 + h_B(f) + d \frac{f^2}{r^2} \right\} (t, r) r^2 dr + \frac{\mu_1^2}{2} \int_{\mathbb{R}} f^2(t, r) r^2 dr = \\
& \int_{\mathbb{R}} J\mu_1 f_t(0, r) f(0, r) r^2 dr + \int_{\mathbb{R}} J \left\{ \frac{J}{2} f_t^2 + \frac{L}{2} f_r^2 + h_B(f) + d \frac{f^2}{r^2} \right\} (0, r) r^2 dr + \frac{\mu_1^2}{2} \int_{\mathbb{R}} f^2(0, r) r^2 dr \\
& - \mu_1 \int_0^t \int_{\mathbb{R}} \left( L f_r^2 + 2d f^2 + a f^2 - \frac{b(d-2)}{d} f^3 + \frac{c(d-1)}{d} f^4 \right) (s, r) r^2 dr ds
\end{aligned} \tag{4.26}$$

Integrating one more time over  $[0, t]$  we further obtain:

$$\begin{aligned}
& \int_{\mathbb{R}} \frac{J\mu_1}{2} f^2(t, r) r^2 dr + \int_0^t \int_{\mathbb{R}} \left\{ J \left( \frac{J}{2} f_t^2 + \frac{L}{2} f_r^2 + h_B(f) + d \frac{f^2}{r^2} \right) + \frac{\mu_1^2}{2} f^2 \right\} (s, r) r^2 dr ds = \\
& \int_{\mathbb{R}} \frac{J\mu_1}{2} f^2(0, r) r^2 dr + t \int_{\mathbb{R}} J\mu_1 f_t(0, r) f(0, r) r^2 dr + t \int_{\mathbb{R}} J \left\{ \frac{J}{2} f_t^2 + \frac{L}{2} f_r^2 + h_B(f) + d \frac{f^2}{r^2} \right\} (0, r) r^2 dr \\
& + t \frac{\mu_1^2}{2} \int_{\mathbb{R}} f^2(0, r) r^2 dr - \mu_1 \int_0^t \int_0^\tau \int_{\mathbb{R}} \left( L f_r^2 + 2d f^2 + a f^2 - \frac{b(d-2)}{d} f^3 + \frac{c(d-1)}{d} f^4 \right) (s, r) r^2 dr ds d\tau
\end{aligned} \tag{4.27}$$

Using the fact that for an arbitrary function  $a \in L^1_{loc}(\mathbb{R}; \mathbb{R})$  we have:  $\int_0^t \int_0^\tau a(s) ds d\tau = \int_0^t (t - \tau) a(\tau) d\tau$  and that  $J, \mu_1, L, c > 0$  we obtain out of the last relation:

$$\int_{\mathbb{R}} \frac{J\mu_1}{2} f^2(t, r) r^2 dr \leq C_1 + C_2 t + C_3 t \int_0^t \int_{\mathbb{R}} f^2(s, r) r^2 dr ds \tag{4.28}$$

which implies for any  $T > 0$  that  $f \in L^\infty(0, T; L^2(\mathbb{R}, r^2 dr))$ . Using this bound and integrating (4.22) on  $[0, T]$  we also get  $f \in L^\infty(0, T; H^1(\mathbb{R}, r^2 dr)) \cap L^\infty(0, T; L^4(\mathbb{R}, r^2 dr))$  and  $f_t \in L^\infty(0, T; L^2(\mathbb{R}, r^2 dr))$ .

**Step 4: the existence of smooth twist solutions** One can easily see that the previously obtained weak solution of equation of the  $f$ -equation (1.22) will provide a weak solution of the  $Q$ -equation (1.20) through the formula  $T(t, x) = f(t, x) \bar{H}(x), \forall t \geq 0, x \in \mathbb{R}^d$ . Due to the weak-strong uniqueness we have that  $T(t, x)$  is also a strong solutions and thus since we can take  $s$  arbitrary we have  $T$  smooth. In particular  $T$  is continuous at 0 which, because of the discontinuity of  $\bar{H}$  necessarily implies that  $f(t, \cdot)$  is continuous on  $[0, \infty)$  for any  $t \geq 0$  and  $f(t, 0) = 0$  hence condition (4.1) holds. Furthermore by evaluating the representation formula  $T(t, x) = f(t, |x|) \bar{H}$  for at the component 11 of the matrices and at the point  $x = (0, r)$  we have that  $T_{11}(t, 0, r) = f(t, r) (\frac{r^2}{r^2} - \frac{1}{3})$  hence  $f(t, r) = \frac{3}{2} T_{11}(t, 0, r)$ . Similarly  $f(t, r) = \frac{3}{2} T_{11}(t, 0, -r)$ . Since  $T_{11}$  is a smooth function, we have that its restriction to the line  $\{(0, r), r \in \mathbb{R}\}$  is also a smooth function that is furthermore even. Thus we get that  $f \in C^\infty[0, \infty)$  with  $f_r(t, 0) = 0$  as a consequence of the evenness of  $T_{11}(t, 0, \cdot)$ . A similar

argument holds for  $f_t$  providing its smoothness and  $f_{rt}(t, 0) = 0$ , hence conditions (4.3) and (4.5) hold. Thus, the arguments provided before the statement of Proposition 1.3 hold in a rigorous sense and we have obtained a twist wave.  $\square$

## 5. Some Open Problems

The study and the techniques developed in the paper generate some natural research directions:

- **Global existence for small data and  $a$  negative** The main technical assumption that we make for obtaining the existence of a global solution is that  $a$  is positive. This captures a physically relevant regime, in which the nematic state is nevertheless just a local but not global minimizer of the bulk potential. It would be interesting and technically challenging to see if one can obtain global existence for  $a$  negative.

We suspect that the case  $a$  negative should be treated with different tools, as that case allows in the case  $J = 0$  for a solution of the  $Q$ -equation (without flow) whose  $L^p$  norms increase (which seems incompatible with our strategy of the proof of global existence).

- **Long-time behaviour, and relation to inertialess version**

The usual expectation for the case of the damped wave equation is that in the long time one has a diffusive-type behaviour and this is shown in a number of papers (see for instance [4]). Our system has a resemblance with a damped wave equation but it is not clear if its structure allows for a similar conclusion.

- **The  $J \rightarrow 0$  limit** It is a natural question to try to understand the singular limit  $J \rightarrow 0$  in order to understand the effect that taking  $J = 0$  has. It is natural to conjecture that after an initial boundary layer in time the solutions will converge strongly to the formal  $J = 0$  limit.
- **More twist-waves (with genuine wave structure)** We provide in the last section an example of a twist-wave solution that will remain a solution even if one formally sets the flow  $u = 0$  and even if one also sets  $J = 0$ . It would be interesting to provide more such examples, and in particular to understand if there are examples which would not survive as twist-wave solutions when setting formally  $J = 0$ .
- **The stability of the twist wave solution** In [3] it was shown that in the stationary case the melting hedgehog solution is stable. This is a crucial feature for determining the physical relevance of such a solution because only the stable solutions can be observed experimentally. It would be thus very interesting to see if one has dynamical stability of the “melting hedgehog wave” solution, i.e. if one starts with the an approximation of the hedgehog initial data in  $Q$  and a small initial data in  $u$  will this solution stay close to the melting hedgehog one? Will it evolve in the long-time to the melting hedgehog wave?

## Appendix

**Lemma 5.1.** *Let  $y$  be a positive function in  $W_{loc}^{1,1}(\mathbb{R}_+)$  and  $x$  a function in  $L_{loc}^1(\mathbb{R}_+)$  that is almost everywhere positive. Let us assume that*

$$y'(t) + x(t) \leq Cy(t)x(t), \quad (5.1)$$

for almost every  $t$  in  $\mathbb{R}_+$ .

There exists  $\varepsilon_0 > 0$  a suitably small number such that if we take the initial datum  $y(0) = y_0 \in (0, \varepsilon_0)$ , then  $y$  and  $x$  belong to  $L^\infty(\mathbb{R}_+)$  and  $L^1(\mathbb{R}_+)$  respectively, and moreover

$$\|y\|_{L^\infty(\mathbb{R}_+)} + \|x\|_{L^1(\mathbb{R}_+)} \leq y_0.$$

*Proof.* Assuming  $y_0 \leq 1/4C$ , we define  $T > 0$  as the sup of  $t > 0$  such that  $y(t) < 1/2C$ . Then, for every  $t \in [0, T]$  we get

$$y'(t) + \frac{1}{2}x(t) \leq 0,$$

so that, integrating from 0 to  $T$ , we deduce

$$y(T) + \frac{1}{2} \int_0^T x(t) \leq y_0 < \frac{1}{2C}.$$

This yields that  $T = +\infty$  and that

$$\|y\|_{L^\infty(\mathbb{R}_+)} + \|x\|_{L^1(\mathbb{R}_+)} \leq y_0.$$

□

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